$\boxed{\text{Lecture 1}}$ From magnetostatics in $\mathbb{R}^3$ to gauge theory

$U \subset \mathbb{R}^3$ open set, in absence of currents the magnetic field $B = B_1 \frac{\partial}{\partial x_1} + B_2 \frac{\partial}{\partial x_2} + B_3 \frac{\partial}{\partial x_3}$

satisfies the Maxwell eqns:

(Gauss) $\operatorname{div} B = 0$ (i.e $\nabla \cdot B = 0$, where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$).

(Ampère) $\operatorname{curl} B = 0$ (i.e $\nabla \times B = 0$)

A point particle is affected via

Lorentz force: $F = q(v \times B)$

charge velocity.

Energy density: $\frac{1}{2}|B|^2$

Natural differential form viewpoint:

Consider $\bar{B} = B_1\, dx_2 \wedge dx_3 + B_2\, dx_3 \wedge dx_1 + B_3\, dx_1 \wedge dx_2$.

two-form in $U \subseteq \mathbb{R}^3$

Then (gauss) is: $d\bar{B} = 0$, i.e $\bar{B}$ is closed.

$\Rightarrow$ locally, $\bar{B}$ is exact, i.e $\bar{B} = d\bar{A}$ for

$\bar{A} = A_1\, dx_1 + A_2\, dx_2 + A_3\, dx_3$ 1-form.

Rmk in physics, one considers

$A = \sum A_i \frac{\partial}{\partial x_i}$ vector potential, $\nabla \times A = B$.

Key point there are many choices for $\bar{A}$!

for any $f: U \to \mathbb{R}$, $\bar{A} + df$ also works.
or anything closed

($A + \text{grad} f$ in vector notation).

$\rightsquigarrow$ first instance of <u>gauge symmetry</u>!

Choosing the right gauge simplifies computations a lot! <u>Ex</u> in terms of A, (Ampère) becomes $\text{grad}(\text{div} A) + \Delta A = 0$

where $\Delta = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}$ ← natural convention for Laplacian.

applied entry-wise.

So if $\text{div} A = 0$ (Coulomb gauge condition)

we simply ask $\Delta A_i = 0$. $(i = 1, 2, 3)$,

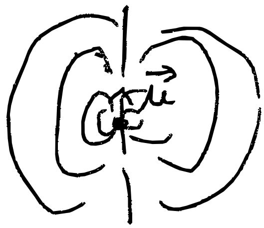
i.e A_i is <u>harmonic</u>.

<u>Example!</u> <u>magnetic dipole</u> in $\mathbb{R}^3 \setminus \{0\}$ with magnetic moment $\vec{\mu}$.

$$A(\vec{x}) = \frac{\mu_0}{4\pi} \cdot \frac{\vec{\mu} \times \vec{x}}{|x|^3} \qquad \left(\frac{x_i}{|x|^3} \text{ is harmonic on } \mathbb{R}^3 \setminus \{0\}, \ \operatorname{div} A = 0. \right)$$

magnetic permeability.

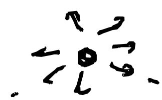
$\Rightarrow$ $B(\vec{x})$ looks like



Of course, $\bar{B}$ is an exact 2-form in $\mathbb{R}^3 \setminus \{0\}$.

<u>Example 2</u> <u>Dirac monopole</u> (1931) of charge g

is $B = \dfrac{\mu_0 g}{4\pi r^2} \hat{r}$ on $\mathbb{R}^3 \setminus \{0\}$



$$\int_{S^2_r} \bar{B} = \mu_0 g \Rightarrow \text{it's non-zero class in } H^2_{dR}(\mathbb{R}^3 \setminus \{0\}) = \mathbb{R}!$$

$\rightsquigarrow$ no global potential.

Key example for us, (open question if it exists in nature).

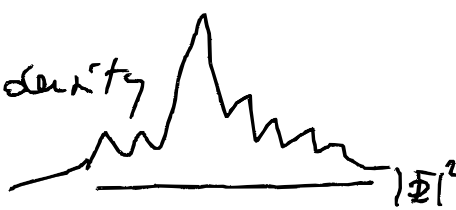
In quantum theory, particles are also described by fields. Simplest case $\Phi : U \to \mathbb{C}$

Klein-Gordon field with charge q, mass $m > 0$ has energy density (set $\hbar = 1$, $c = 1$)

$$\mathcal{L} = \frac{1}{2} |\nabla \Phi - iq \vec{A} \Phi|^2 + \frac{1}{2} m |\Phi|^2 .$$

$\mathbb{C}$-valued vector.

Here $|\Phi|^2$ is the probability density of the particle,



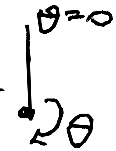
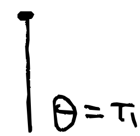
unchanged by $\Phi \rightsquigarrow e^{i\chi} \Phi$ for $\chi : U \to \mathbb{R}$. But $\mathcal{L}$ is not!

The gauge symmetry leaving $\mathcal{L}$ invariant in this case is

$$A \mapsto A + \operatorname{grad} f, \quad \Phi \mapsto e^{iqf}\,\Phi.$$

<u>Dirac</u>: this has topological meaning!

For the Dirac monopole, write

$\mathbb{R}^3 \setminus \{0\} = U^0 \cup U^\pi$ $\quad U^0 =$  $\quad U^\pi =$ 

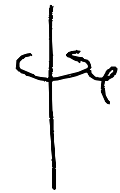
Then have $H^2_{dR} = 0 \Rightarrow B$ has primitive.

Explicitly (set $\mu_0 = 1$)

$$A^0 = -\frac{g}{4\pi}\frac{1}{r\sin\theta}(1+\cos\theta)\,\hat{\varphi}$$ 

$$A^\pi = \frac{g}{4\pi}\frac{1}{r\sin\theta}(1-\cos\theta)\,\hat{\varphi}$$

On $U^0 \cap U^\pi$, $\bar{A}^0 = \bar{A}^\pi + d\left(-\frac{g}{2\pi}\varphi\right)$



(well-defined closed form!).

If Φ is KG field of charge q, well-definedness

$\Rightarrow \frac{qg}{2\pi} \in \mathbb{Z} \Rightarrow q$ is quantized.

Rmk this is closely related to building

non trivial vector bundles on S^n

via clutching functions: $\mathrm{Vect}_k^{\mathbb{C}}(S^n) \overset{1:1}{\longleftrightarrow} \pi_{n-1}(GL(k,\mathbb{C}))$

First goal: setup the geometric language to describe gauge theory in general (Weyl, Yang, etc...), generalizing magnetostatics ($= U(1)$-gauge theory).

$U(1) = \{e^{i\theta}\}$.

Basic dictionary on M manifold

field Φ $\rightsquigarrow$ section of vector bundle
$$E \to M.$$

potential A $\rightsquigarrow$ connection A on E

$\nabla - iqA$ $\rightsquigarrow$ covariant derivative ∇_A on E

magnetic field B $\rightsquigarrow$ curvature F_A of A

quantization of charge $\rightsquigarrow$ characteristic classes of E.

gauge symmetry $\rightarrow$ bundle automorphisms

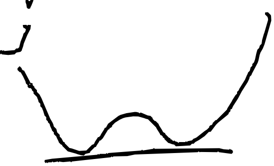
$\rightarrow$ we'll study fundamental objects which appear both in physics (e.g. standard model) but also have intrinsic geometric topological meaning.

<u>Key example</u> $A=\sum A_i\,dx_i$, $\Phi : U \to \mathbb{C}$

as before (B curvature 2-form).

The <u>abelian Yang-Mills-Higgs</u> energy (for a fixed parameter $\lambda>0$) is

$$\frac{1}{2}|B|^2+\frac{1}{2}|(\nabla-iqA)\Phi|^2+\underbrace{\frac{\lambda}{8}(1-|\Phi|^2)^2}_{\text{Higgs potential}}$$



aka <u>gauged Landau-Ginzburg</u> energy

describes the behaviour of superconducting

materials. Physically, the cases $\lambda<1$ and

$\lambda>1$ lead to different interesting phenomena;

Mathematically, $\lambda=1$ (→ critical coupling) (e.g. alloy of

lead and 1-2% thallium) is

is the most interesting, and is in fact (10)

the reason why in Heegaard Floer homology

(key tool in contemporary low-dimensional top)

one studies the symmetric products of a

Riemann surface. We'll spend the rest of

the semester studying and explaining the

geometry, topology & analysis behind this.

Geometric setup M smooth mfld (usually cpt, or $\mathbb{R}^4$).

$E \to M$ vector bundle (either $\mathbb{C}$ or $\mathbb{R}$)

Connection A on E is defined through

covariant derivative

$$\underset{\substack{\shortparallel \\ d_A}}{\nabla_A} : \Omega^0(E) \to \Omega^1(E) \qquad \left(\begin{array}{c} \Omega^p(E) = E\text{-valued} \\ p\text{-forms on } M \end{array} \right)$$

Satisfying Leibniz rule:

$$\nabla_A(f\cdot s) = df\cdot s + f\nabla_A s \quad \forall f\in C^\infty(M)$$
$$s\in \Omega^0(E).$$

{Connection on E} is affine space / $\Omega^1(\mathrm{End}(E))$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathfrak{gl}(E)$

We'll think of bundles equipped with extra structures, based on some Lie group G with symmetries (with Lie algebra $\mathfrak{g}$) $\rightsquigarrow$ "G-vector bundle". Ex:

- no structure ($G = GL(n;\mathbb{C})$ or $GL(n;\mathbb{R})$).

- E Hermitian, structure $\langle\,,\,\rangle$. For unitary
  $G = U(n) = \{A \mid A^*A = \mathrm{Id}_n\}$ unitary group
  $\mathfrak{g} = \mathfrak{u}(n) = \{$skew-Hermitian matrices $B^* = -B\}$.

- E special unitary, structure $\langle\,,\,\rangle$ + $\det E \cong \mathbb{C}$ trivialization. $G = SU(n) \leftarrow \det = 1$ in $U(n)$

$su(n) = \{$traceless skew-Hermitian$\}$.

•) oriented euclidean (real), $G = SO(n)$

special orthogonal group, $so(n) = \{$antisymmetric$\}$.

We will consider connections A compatible with the extra structure. E.g. for $\langle , \rangle$, want

$d\langle s, s'\rangle = \langle d_A s, s'\rangle + \langle s, d_A s'\rangle \quad \forall s, s' \in \Omega^0(E)$,

These form a smaller space, affine / $\Omega^1(\mathfrak{g}_E)$

Ex $E \to M$ unitary,

$\{$unitary connections on $E\}$ affine / $\Omega^1(U(E))$

where $U(E) = \left\{ \begin{array}{c} E \xrightarrow{\ell} E \\ \searrow M \swarrow \end{array} \;\middle|\; \ell \text{ fibrewise skew-hermitian} \right\}$.

Rmk In general, one uses the language of principal G-bundles & associated vector bundles.

<u>Locally</u>, E is trivial so

$E \overset{\tau}{\cong} U \times \mathbb{C}^n \quad (U \subseteq \mathbb{R}^n)$ so we can write

$\nabla_A = d + A^\tau$, $\quad A^\tau \in \Omega^1(\mathfrak{g})$ (= matrix of 1-forms)

If $\{x_i\}$ are coordinates in U, we write

$\nabla_A = \sum \nabla_i \, dx_i$ with

$\nabla_i = \frac{\partial}{\partial x_i} + A_i^\tau$, $\quad A_i^\tau$ function with values in $\mathfrak{g}$.

$\boxed{\text{Lecture 2}}$ Connections & topology.

$E \to M$ vector bundle with structure G.

$\Rightarrow \mathcal{A} = \{A \mid A \text{ compatible connections on } E\}$

$\hookrightarrow$ affine space / $\Omega^1(\mathfrak{g}_E)$

<u>Gauge transformations</u> are

$$\mathcal{G} = \left\{ u \;\middle|\; \begin{array}{c} E \xrightarrow{u} E \\ \searrow_M \swarrow \end{array} \begin{array}{l} \text{bundle automorphism} \\ \text{preserving structure} \end{array} \right\}.$$

(= symmetries of E)

Of course, $\mathcal{G} \curvearrowright \Gamma(E)$ by multiplication;
it also acts on $\mathcal{A}$ by pullback, i.e.

$$\nabla_{u \cdot A} s = u \nabla_A (u^{-1} s) \quad \forall A \in \mathcal{A},\ s \in \Gamma(E)$$

$$\rightsquigarrow \nabla_{u \cdot A} = \nabla_A - \underbrace{(\nabla_A u) u^{-1}}_{\in \Omega^1(\mathfrak{g}_E)}$$

Here, we thing $u \in \mathcal{G} \in \Omega^0(\mathrm{End}(E))$, and use the fact: ∇_A on E naturally induces connection of $\mathrm{End}(E) = E \otimes E^*$ (still called ∇_A), given in a trivialization τ (so $\nabla_A = d + \underset{\in \Omega^1(g)}{A^\tau}$ on E) by

$$d + [A^\tau, -] \quad \text{on } \mathrm{End}(E).$$

So locally, in the trivialization $u\tau$

$$A^{u\tau} = A^\tau - \{(d + [A^\tau, -])u\} u^{-1} =$$

$$= A^\tau - \{du + A^\tau u - u A^\tau\} u^{-1} =$$

$$= u A^\tau u^{-1} - (du) u^{-1} \quad \in \Omega^1(g).$$

Example $L \to M$ hermitian line bundle $(G = U(1))$

If $u = e^{if}$, $f : M \to \mathbb{R}$, then

$dv = ie^{if}(df)$

$\Rightarrow A^{vz} = A^{z} - i\,df$, same thing from magnetostatic!

<u>Rmk</u> $\mathcal{G} = \{ v : M \to U(1) \}$ gauge group.

$\Rightarrow \pi_0(\mathcal{G}) = H^1(M;\mathbb{Z}) \quad (U(1) = S^1 = K(\mathbb{Z},1))$

$\Rightarrow$ not all gauge transformation are $v = e^{if}$.

Natural object to study is

$\mathcal{A}/\mathcal{G}$ $\quad \dfrac{\text{moduli space of connections}}{\text{(up to gauge)}}$,

really, we are interested in gauge-independent quantities, i.e well-defined functions

$\mathcal{L} : \mathcal{A}/\mathcal{G} \to \mathbb{R}$

These naturally arise from the geometry of the connection, in particular its <u>curvature</u>. (This intuitively measures how covariant derivatives

do <u>not</u> commute:)

Algebraically, we can obtain

$$d_A : \Omega^p(E) \to \Omega^{p+1}(E) \quad \text{(covariant exterior derivative)}$$

Combining: •) $d_A = \nabla_A : \Omega^0(E) \to \Omega^1(E)$

•) $d_A(\alpha \wedge \beta) = d_A\alpha \wedge \beta + (-1)^p \alpha \wedge d_A \beta$

$$\forall \alpha \in \Omega^p(E),\ \beta \in \Omega^{p+1}(E).$$

Key point:

$$\Omega^0(E) \xrightarrow{d_A} \Omega^1(E) \xrightarrow{d_A} \Omega^2(E) \xrightarrow{d_A} \Omega^3(E) \to \dots$$

is <u>not</u> a complex!

Checking C^∞-linearity, we have

<u>Lemma</u> $\exists\, F_A \in \Omega^2(\mathfrak{g}_E) \subseteq \Omega^2(\mathrm{End}(E))$ s.t

$d_A d_A \alpha = F_A \cdot \alpha \quad \forall \alpha \in \Omega^p(E)$

wedge and $\mathrm{End}(E) \otimes E \to E$.

· F_A is the <u>curvature</u> of A.

<u>Rmk</u> (M,g) Riemannian $\Rightarrow \nabla_{LC}$ on TM

$so(n)$ connection

$\Rightarrow F_{LC} \in \Omega^2(so(TM))$ Riemann curvature tensor

4-indices, antisymmetry properties.

<u>Locally</u> $\nabla_A = d + A^\tau$, $A^\tau \in \Omega^1(\mathfrak{g})$

Then $F_A^\tau = dA^\tau + A^\tau \wedge A^\tau \quad (=: \Omega \in \Omega^2(\mathfrak{g}))$

wedge & matrix multiplication.

In explicit coordinates $\{x_i\}$,

$F_A^{\tau} = \sum F_{ij}\, dx_i \wedge dx_j$ with

$$F_{ij} = [\nabla_i, \nabla_j] = \left[\frac{\partial}{\partial x_i} + A_i^{\tau}, \frac{\partial}{\partial x_j} + A_j^{\tau}\right] =$$

$$= \frac{\partial}{\partial x_i} A_j^{\tau} - \frac{\partial}{\partial x_j} A_i^{\tau} + [A_i^{\tau}, A_j^{\tau}]$$

$\hookrightarrow$ quadratic term!

Notice that if $G = U(1) \Rightarrow A \in i\Omega^1$ and

$F_A = dA$ is simply the magnetic field $B \in i\Omega^2$!

Gauss' law $dB = 0$ is a special case of

<u>Lemma</u> (Bianchi identity) $d_A F_A = 0$ ($\in \Omega^3(\mathfrak{g}_E)$)

<u>Proof</u> Locally, this is just

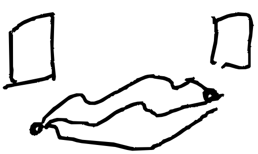
$d\Omega + [A^{\tau}, \Omega] = 0$, which follows from

differentiating $\Omega = dA^{\tau} + A^{\tau} \wedge A^{\tau}$ $\square$.

<u>Lemma</u> $u \in \mathcal{G}$, then $F_{u \cdot A} = u F_A u^{-1} \in \Omega^2(\mathfrak{g}_E)$.

So curvature is <u>not</u> gauge invariant

<u>Notice</u> Being flat is a gauge invariant notion! Also flat $\Rightarrow$ covariant derivatives commute $\Rightarrow$ parallel transport is htpy invariant



so we obtain well defined

$\{$flat connections on $E\}/\text{gauge} \longrightarrow \{\pi_1(M) \to G\}/G$ acting by conj.

<u>holonomy representation</u> $\pi_1(M) \curvearrowright E_{x_0}$.

Of course, if E is trivial ($E \cong M \times \mathbb{C}^n$ G-iso)

$\Rightarrow \exists$ flat connection.

Conversely, one can use curvature to detect topological non triviality

$\rightsquigarrow$ Chern-Weil approach to <u>characteristic classes</u>.

<u>Key point</u> $G \curvearrowright \mathfrak{g}$ by conjugation, invariant functions $\mathfrak{g} \to \mathbb{R}$ or $\mathbb{C}$ applied to F_A give gauge invariant quantities of A.

<u>Ex</u> $F_A \in \Omega^2(\mathrm{End}\, E)$ ← $n \times n$ matrix of 2-forms / matrix valued 2-form.

$\Rightarrow \mathrm{tr}\, F_A \in \Omega^2_{\mathbb{C}}$ is gauge invariant!

<u>Rmk</u> $\mathrm{tr}\, F_A \in i\Omega^2$ if A is unitary;

Similarly, we can consider $\det F_A \in \Omega^{2n}_{\mathbb{C}}$.

We'll focus on the $G = GL_n(\mathbb{C})$ case.

<u>Fact</u> $GL_n(\mathbb{C})$-invariant polynomials on $gl_n(\mathbb{C})$ are polynomials in symmetric functions of eigenvalues; assemble for $L \in gl_n(\mathbb{C})$ as

$$\det(1+qL) = \sum q^k \operatorname{tr}(\Lambda^k L)$$

with q formal variable.

<u>Prop</u> P invariant polynomial (e.g. $P = \operatorname{tr}, \det$) of degree k, then $P(F_A) \in \Omega^{2k}_{\mathbb{C}}$ is closed & $[P(F_A)] \in H^{2k}(M;\mathbb{C})$ is independent of the connection A

$\Rightarrow$ it's a topological invariant of the bundle!

<u>Proof</u> can study $\log\det(1+qL)$ formally as power series in q.

Let's see what happens along a path $A(t)$ of connections, let's do local computation $A^2(t) = A(t) \in \Omega^1(\mathfrak{gl}_n(\mathbb{C}))$

$\Omega = dA + \underbrace{A \wedge A}_{A \cdot A}$ so differentiating in t

$\dot{\Omega} = d\dot{A} + A\dot{A} + \dot{A}A$

Formally (as power series in q)

$$\frac{d}{dt} \log \det(1 + q\Omega) = q \,\mathrm{tr}\left[(1+q\Omega)^{-1}\dot{\Omega}\right]$$

$$= \sum_{\ell=0}^{\infty} (-1)^{\ell} q^{\ell+1} \mathrm{tr}(\Omega^{\ell}(d\dot{A} + A\dot{A} + \dot{A}A)).$$

Now $\mathrm{tr}(\Omega^{\ell}(A\dot{A} + \dot{A}A)) \overset{\text{cyclic symmetry of trace}}{=}$

$= \mathrm{tr}(\Omega^{\ell}A\dot{A} - A\Omega^{\ell}\dot{A}) \underset{\text{Bianchi identity.}}{=} \mathrm{tr}((d\Omega^{\ell})\dot{A})$

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$\Rightarrow \frac{d}{dt} \log\det(1+q\Omega) = d\left(\sum_{l=0}^{\infty}(-1)^{l} q^{l+1} \operatorname{tr}(\Omega^{l}\dot{A})\right)$

is <u>exact</u>. So deforming the connection changes it by an exact form.

•) locally, consider $A(t) = tA$

$\Rightarrow$ locally exact hence closed.

•) globally, $\mathcal{A}$ connected $\Rightarrow$ cohomology class independent of connection $\square$.

<u>Fact</u> $P(L) = \left(\frac{i}{2\pi}\right)^{k} \operatorname{tr}(\Lambda^{k} L)$ determines

$c_k(E) \in H^{2k}(M;\mathbb{Z}) \subseteq H^{2k}(M;\mathbb{C})$

k-th <u>chern class</u>.

<u>Rmk</u> Dirac monopole $\rightsquigarrow U(1)$-bundle. corresponds to generator in $H^{2}(\mathbb{R}^{3}\setminus\{0\};\mathbb{Z}) \cong \mathbb{Z}$.

These satisfy nice <u>axioms</u>!

- $c_0(E) = 1 \in H^0$, $c_i(E) = 0$ if $i > \dim_{\mathbb{C}} E$.

- $f: N \to M$, $E \to M$ vector bundle
  $\Rightarrow f^*E \to N$ pullback bundle has
  $c_k(f^*E) = f^* c_k(E)$ (naturality)

- $E, F \to M$, then
  $c_k(E \oplus F) = \sum c_i(E) \cup c_{k-i}(F)$ (Whitney sum formula)

- Normalization: $\gamma \to \mathbb{CP}^1$ tautological bundle has $c_1(\gamma) = -1 \in H^2(\mathbb{CP}^1; \mathbb{Z}) \cong \mathbb{Z}$
  natural orientation.

These uniquely determine the Chern classes!

Lecture 3 Yang-Mills theory on Riemannian manifolds.

Recall energy density of magnetic field B in $\mathbb{R}^3$ is $\frac{1}{2}|B|^2$. This generalizes to

$E \to M$ G-bundle, A compatible connection

$\rightsquigarrow F_A \in \Omega^2(\mathfrak{g}_E)$ curvature 2-form.

$(u \in \mathcal{G} \Rightarrow F_{u\cdot A} = u F_A u^{-1})$.

To define its norm $|F_A|$

•) g Riemannian metric on M.

•) G-invariant norm $|\cdot| : \mathfrak{g} \to \mathbb{R}$

(induced by inner product).

Ex.) $\mathfrak{g} = \mathfrak{u}(n) = \{$skew symmetric matrices$\}$.

$|\xi|^2 := L^2$-norm as vector in $\mathbb{C}^{n \times n} = -\operatorname{Tr}(\xi^2)$

is $U(n)$ invariant.

•) in general such norm exists for G cpt

by averaging.

Def The Yang-Mills functional on (M, g) is

$$YM(A) = \frac{1}{2} \int_M |F_A|^2 \, dvol_g,$$

descends to $\mathcal{A}/\mathcal{G} \xrightarrow{YM} \mathbb{R}$

Rmk introduced by Yang & Mills (1954) for

$G = SU(2)$ to describe weak interactions;

the three components of $F_A \in \Omega^2(su(2))$

are called W^+, W^-, Z^0 gauge bosons.

Q what are the critical points of this functional?

The answer involves a non-linear version of Hodge theory, which we now review.

Recall M^n smooth mfld, de Rham complex

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \to \cdots \xrightarrow{d} \Omega^n(M)$$

with cohomology $\mathrm{Ker}\, d / \mathrm{Im}\, d \cong H^i(M; \mathbb{R})$.

No preferred representative in a given cohomology class!

If (M,g) is cpt, $\exists$ natural L^2-norm

$$\|\alpha\|^2_{L^2} = \int_M |\alpha|^2 \, d\mathrm{vol}_g \qquad \alpha \in \Omega^i(M).$$

(induced by inner product $\langle \alpha, \beta \rangle_{L^2} = \int_M \langle \alpha, \beta \rangle \, d\mathrm{vol}_g$)

$\Rightarrow$ it's natural to consider the form that minimizes in a given class.

<u>Problem</u> does it exist, and is it unique?

$\rightsquigarrow$ Hodge theory.

<u>Def</u> $(\overset{\to \dim_{\mathbb{R}} = n}{V}, \langle , \rangle, o)$ oriented Euclidean v. space.

$\Rightarrow \ * : \Lambda^i V \to \Lambda^{n-i} V$ <u>Hodge star</u>

sends for oriented ON basis $e_1, \dots e_n$

$*(e_1 \wedge \dots \wedge e_k) = e_{k+1} \wedge \dots \wedge e_n$.

<u>Eg</u> in $(\mathbb{R}^3)^*$, $*dx_1 = dx_2 \wedge dx_3$, $*dx_2 = dx_3 \wedge dx_1$

<u>Lemma</u> •) $\alpha, \beta \in \Lambda^i V \Rightarrow \alpha \wedge *\beta = \langle \alpha, \beta \rangle \underbrace{dvol}_{e_1 \wedge \dots \wedge e_n}$.

•) $*^2 : \Lambda^i V \circlearrowleft$ is $(-1)^{i(n-i)}$

On (M^n, g, o), orientation, this induces

$$* : \Omega^i(M) \to \Omega^{n-i}(M).$$

We have $d : \Omega^{p-1} \to \Omega^p$, metric independent

<u>Def</u> $d^* : \Omega^p \to \Omega^{p-1}$ is given by

$$d^* = (-1)^{n(p+1)+1} * d *$$

Assume from now on <u>M cpt</u>, so that Ω^i are inner product spaces.

<u>Key Lemma</u> d^* is the (formal L^2) adjoint of d,

i.e $\langle d\alpha, \beta \rangle_{L^2} = \langle \alpha, d^*\beta \rangle_{L^2}$

$$\forall \alpha \in \Omega^{p-1}, \beta \in \Omega^p.$$

<u>Proof</u> integration by parts, i.e stokes!

$\alpha \wedge *\beta \in \Omega^{n-1}$, compute pointwise

$$d(\alpha \wedge *\beta) = d\alpha \wedge *\beta + (-1)^{p-1} \alpha \wedge d*\beta \overset{!}{=}$$

check signs!

$$= d\alpha \wedge *\beta - \alpha \wedge *d^*\beta = \left(\langle d\alpha, \beta\rangle - \langle \alpha, d^*\beta\rangle\right) dvol.$$

□

Rmk For functions on $\mathbb{R}$ (cpty supported)

$$\int \left(\frac{d}{dx} f\right) g = - \int f \left(\frac{d}{dx} g\right),$$ so $\frac{d}{dx}$ is skew-adjoint.

Now assume $\alpha_0 \in \Omega^p$ closed minimizes L^2-norm in $[\alpha_0]$. Then $\forall \eta \in \Omega^{p-1}$

$$0 = \frac{d}{dt}\Big|_{t=0} \|\alpha_0 + t d\eta\|_{L^2}^2$$

$$= \frac{d}{dt}\Big|_{t=0} \left(\|\alpha_0\|_{L^2}^2 + 2t\langle \alpha_0, d\eta\rangle_{L^2} + t^2 \|d\eta\|_{L^2}^2\right)$$

$$= 2\langle \alpha, d\eta\rangle = 2\langle d^*\alpha, \eta\rangle$$

$\Rightarrow \quad d^* \alpha_0 = 0 \qquad$ (i.e α is coclosed).

So $\alpha_0 \in \ker(d + d^* : \Omega^p \to \Omega^{p+1} \oplus \Omega^{p-1})$.

Def The Hodge Laplacian on (M, g) is

$$\Delta = (d + d^*)^2 : \Omega^p \circlearrowleft$$
$$\quad = dd^* + d^* d$$

Basic facts •) on Ω^0, Δ is the usual

Laplace-Beltrami $\Delta = -\operatorname{div} \operatorname{grad} f$

$$\left(\begin{tikzcd} \Omega^0 \arrow[r, shift left, "d"] \arrow[rr, bend left, "\mathrm{grad}"] & \Omega^1 \arrow[l, shift left, "d^*"] \arrow[r, phantom, "\cong"] & \Gamma(TM) \arrow[ll, bend left, "-\mathrm{div}"] \end{tikzcd} \right).$$

•) Δ is (formally L^2) self adjoint,

$$\langle \alpha, \Delta \beta \rangle_{L^2} = \langle \Delta \alpha, \beta \rangle_{L^2}.$$

•) Δ is non-negative, indeed

$$\langle \alpha, \Delta\alpha \rangle_{L^2} = \|(d+d^*)^2 \alpha\|_{L^2}^2 \geq 0.$$

This also shows $\ker \Delta = \ker(d+d^*)$!
= harmonic p-forms.

So we were asking for harmonic representatives!

<u>Hodge theorem</u> There is an L^2-orthogonal decomposition for (M,g) cpt

$$\Omega^p = \underbrace{d\Omega^{p-1}}_{\text{exact}} \oplus \underbrace{\mathcal{H}^p}_{\text{harmonic}} \oplus \underbrace{d^*\Omega^{p+1}}_{\text{coexact}}$$

and $\mathcal{H}^p := \ker \Delta = (\operatorname{Im}\Delta)^\perp \cong H^p(M;\mathbb{R})$

This theorem provides a deep link between geometry, topology, and linear PDEs; we'll sketch a proof in a few lectures!

<u>Rmk</u> Poincaré duality is $*: \mathcal{H}^p \to \mathcal{H}^{n-p}$.

Back to Yang-Mills on (M,g), $E \to M$,

story is very similar; let's see how

$YM(A) = \frac{1}{2}\int |F_A|^2 dvol_g = \frac{1}{2}\|F_A\|_{L^2}^2$ changes

<u>Lemma</u> For $a \in \Omega^1(\mathfrak{g}_E)$, we have

$$F_{A+a} = F_A + d_A a + a \wedge a$$

<u>Rmk</u> In the $U(1)$ case, this is just

$F_{A+a} = F_A + da$, same computation as earlier!

<u>Proof</u> $F_A \cdot s = d_A d_A s$. by def (d_A on $\Omega^i(E)$)

$\Rightarrow F_{A+a} \cdot s = d_{A+a} d_{A+a} s =$

$= d_{A+a}(d_A s + as) =$

$$= d_A d_A s + a \wedge d_A s + \overbrace{(d_A a)s - a \wedge d_A s}^{d_A(as)} + (a \wedge a)s.$$

$\square$

So $YM(A+ta) = \frac{1}{2}\|F_{A+ta}\|^2_{L^2} =$

$$= \frac{1}{2}\|F_A\|^2_{L^2} + t\langle d_A a, F_A\rangle_{L^2} + O(t^2).$$

$$\Rightarrow \left.\frac{d}{dt}\right|_{t=0} YM(A+ta) = \langle d_A a, F_A\rangle_{L^2} =$$

$$= \langle a, d_A^* F_A\rangle_{L^2},$$

where $d_A^* : \Omega^2(\mathfrak{g}_E) \to \Omega^1(\mathfrak{g}_E)$ adjoint of d_A.

$\Rightarrow$ critical points (called YM connections)

$$\boxed{d_A^* F_A = 0} \quad \text{Yang–Mills equations,}$$

2nd order <u>nonlinear</u> PDE in A.

(reduces to Ampère eqns in magnetostatics).

Bianchi: $d_A F_A = 0$

$\rightsquigarrow$ so we are looking at some non-linear version of $d + d^*$! Indeed simplest case:

<u>Prop</u> $L \to (M, g)$ hermitian line bundle on cpt M.
$\Rightarrow \exists$ Yang-Mills connection A.

<u>Proof</u> Fix A_0 connection, $F_{A_0+a} = F_{A_0} + da$.

$a \in i\Omega^1$. Want to solve,

$$0 = d^*(F_{A+a}) = d^* F_A + d^* da .$$

By Hodge theorem, $\Delta : d^*\Omega^2 \circlearrowleft$ isomorphism

$\Rightarrow d^* F_A = \Delta b$ for some $b \in d^* \Omega^2$

$= d^* db \qquad (d^* b = 0)$

$\Rightarrow$ choose $a = -b$. $\square$.

The case G non-abelian is much more complicated. We'll see next time interesting things happen when $\dim M = 4$!

In <u>dim 2</u>, $d_A^*: \Omega^2(g_E) \to \Omega^1(g_E)$ is $-*d_A*$;

but $*F_A \in \Omega^0(g_E)$ so YM eqns say:

$*F_A$ is A-<u>covariantly constant</u>

Recall in trivialization τ, $d_A \equiv d + [A^\tau, -]$;

so if $*F_A = x \in Z(\mathfrak{g}) \Rightarrow A$ is Yang-Mills.

constant, central curvature (well-defined!) generalizes flat!

<u>Ex</u> $G = U(n)$ $\quad Z(U(n)) = i\mathbb{R} \cdot I$

this determines $\pi_1(M) \to PU(n) = U(n)/_{\text{center}}$

$\text{center} = U(1) \cdot \mathrm{Id}$

projective unitary representation.

$\boxed{\text{Lecture 4}}$ Instantons, monopoles & vortices.

Let's study $YM(A) = \frac{1}{2}\|F_A\|^2_{L^2}$

non-abelian cpt Lie group

$SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar\beta \\ \beta & \bar\alpha \end{pmatrix} \,\middle|\, |\alpha|^2+|\beta|^2=1 \right\} \equiv S^3$ ↪ unit quaternions.

with invariant norm $\langle a,b\rangle = \mathrm{Tr}(ab^*)$ on

$su(2)$ = traceless-skew Hermitian $(\equiv \mathrm{Im}\,\mathbb{H}|^3)$

$= \mathrm{span}_{\mathbb{R}} \left\langle \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \right\rangle$ ↪ Pauli matrices σ_i.

on $(\mathbb{R}^4, g_{flat})$ or (M^4, g) cpt. Three special things happen!

I) invariance under conformal rescaling $g \mapsto e^{2f} g$. (M cpt).

Explicitly in $\mathbb{R}^d$ $\nabla_A = d + A$ ↪ $\in \Omega^1(su(2))$

rescaling by μ acts as

$$A \mapsto A^{(\mu)} \text{ with } A^{(\mu)}(x) = \mu A(\mu x)$$

$\Rightarrow F_{A^{(\mu)}}(x) = \mu^2 F_A(\mu x)$. For finite energy,

$$\Rightarrow YM(A^{(\mu)}) = \int |F_{A^{(\mu)}}(x)|^2 dvol = \mu^4 \int |F_A(\mu x)|^2 dvol$$

$$= \mu^{4-d} \int |F_A|^2 dx.$$

change of variables

In $\mathbb{R}^d$ we can have critical points only for $d=4$!

$\underline{\text{II}}$) <u>Self-duality</u>: on (M^{4k}, g),

$$* : \Omega^{2k}(M) \circlearrowleft, \quad *^2 = 1$$

$$\Rightarrow \underbrace{\Omega^{2k} = \Omega^+}_{\text{self-dual}} \oplus \underbrace{\Omega^-}_{\text{antiself-dual}} \quad (\pm 1)\text{-eigenspaces.}$$

On M^4, $\Omega^+ = \text{span}\left\{ \begin{array}{l} dx_1 dx_2 + dx_3 dx_4 \\ dx_1 dx_3 + dx_4 dx_2 \\ dx_1 dx_4 + dx_2 dx_3 \end{array} \right\}$.

In 4D, $\Omega^2 = \Omega^+ \oplus \Omega^-$ corresponds

to the exceptional decomposition

$$so(4) = so(3) \oplus so(3) \quad (\text{as } so(4)\text{-reps});$$

in gauge theory, this leads to

$$F_A = F_A^+ + F_A^- \in \Omega^2(g_E) = \Omega^+(g_E) \oplus \Omega^-(g_E).$$

pointwise orthogonal, so $F_A^+ \wedge F_A^- = 0$ pointwise.

III) Topological bounds on (M, g) cpt:

By Chern-Weil we have

$$\int_M \mathrm{Tr}(F_A \wedge F_A) = 8\pi^2 c_2(E) \in H^4(M;\mathbb{R}) \cong \mathbb{R}.$$

$\Omega^4(su(2))$

$\Big(L \in su(2)$ is conjugate to $\begin{pmatrix} i\lambda & 0 \\ 0 & -i\lambda \end{pmatrix}$ $\lambda \in \mathbb{R}$

$\Rightarrow \mathrm{Tr}(L^2) = -2\lambda^2 = -2\det(L). \Big)$

The YM functional can be written as

$$YM(A) = \frac{1}{2}\int |F_A|^2 \, dvol = -\frac{1}{2}\int \mathrm{Tr}(F_A \wedge \underbrace{*F_A}_{\in \Omega^2(\mathfrak{g}_E)}).$$

Hence we have

$$\int \mathrm{Tr}(F_A \wedge F_A) = \int \mathrm{Tr}(F_A^+ \wedge F_A^+) + \int \mathrm{Tr}(F_A^- \wedge F_A^-)$$

$$= \int \mathrm{Tr}(F_A^+ \wedge * F_A^+) - \int \mathrm{Tr}(F_A^- \wedge * F_A^-)$$

$$= -\|F_A^+\|^2_{L^2} + \|F_A^-\|^2_{L^2}$$

Using $YM(A) = \frac{1}{2}\left(\|F_A^+\|^2_{L^2} + \|F_A^-\|^2_{L^2}\right)$, we obtain

<u>Thm</u> (Bogomolnyi bound)

$$YM(A) = \|F_A^+\|^2_{L^2} + 4\pi^2 c_2(E).$$

purely topological quantity.

<u>Cor</u> Assume $c_2(E) \geq 0$. Then

$YM(A) \geq 4\pi^2 c_2(E)$, and minimum realized

exactly for $\boxed{F_A^+ = 0}$ <u>antiself duality eqns</u>. (ASD)

This is a <u>first order</u>, gauge & conformal invariant equation.

Of course, it implies Yang-Mills $d_A^* F_A = 0$.

(Indeed, $0 \underset{\text{Bianchi}}{=} d_A F_A \underset{\text{ASD}}{=} d_A(-*F_A)$).

<u>BPST 1975</u> there are interesting finite energy ASD connections in $\mathbb{R}^4$!

<u>Rmk</u> not possible in abelian (linear) theory! (L^2 harmonic form vanishes)

Use action $SO(4) \curvearrowright \mathbb{R}^4$, $SO(4) \curvearrowright so(3) \cong su(2)$

$(SU(2) \xrightarrow{2:1} SO(3))$, impose equivariance on A

$\Rightarrow$ ASD reduces to explicitly solvable first order ODE.

Identify $\mathbb{R}^4 \equiv \mathbb{H}$, $su(2) = \mathrm{Im}\,\mathbb{H}$ then

$$A(x) = \mathrm{Im}\left(\frac{x\, d\bar{x}}{1+|x|^2}\right) \in \Omega^1(su(2))$$

$$F_A(x) = \frac{dx \wedge d\bar{x}}{(1+|x|^2)^2} \in \Omega^-(su(2))$$

$\Rightarrow$ BPST (or basic) instanton

with $YM(A) = 4\pi^2$ so "$c_2 = 1$".

To interpret this topologically, if A is any connection with $|F_A| \to 0$ sufficiently fast,

$\Rightarrow$ A is asymptotically flat

$$\Rightarrow A \simeq -(du)u^{-1} \text{ for } u : \underset{\substack{\simeq \\ S^3}}{\mathbb{R}^4 \setminus B_R} \to \overset{\substack{S^3 \\ \shortparallel}}{SU(2)}$$

$\Rightarrow$ well defined $\deg_\infty(A) \in \mathbb{Z}$.

(corresponding to $\pi_3(S^3) = \mathbb{Z}$) and

$$YM(A) = \|F_A^+\|_{L^2}^2 + 4\pi^2 \deg_\infty(A).$$

This is an example of a <u>topological soliton</u>

<u>Other source</u>: (M,g) Riemannian, $E \to M$ G-bundle.

$\mathcal{C} = \{(A, \Phi) \mid A$ compatible connection, Φ section$\}$.

"Higgs field"

For fixed $\lambda > 0$

$$YMH(A,\Phi) := \frac{1}{2}\int |F_A|^2 + \frac{1}{2}|\nabla_A \Phi|^2 + \frac{\lambda}{8}\int (1-|\Phi|^2)^2$$

<u>Yang-Mills-Higgs functional</u>, invariant

for $\mathcal{G} \curvearrowright \mathcal{C}$. $\left(\nabla_{u\cdot A}(u\cdot\Phi) = u\cdot\nabla_A\Phi \right)$

by def of $u\cdot A$

Rmk name comes from $G = SU(2)$, introduced to explain mass of bosons in weak interaction. (Higgs mechanism). When $G = U(1)$, this is describes superconductivity (Ginzburg-Landau)

The same rescaling argument in $\mathbb{R}^d$

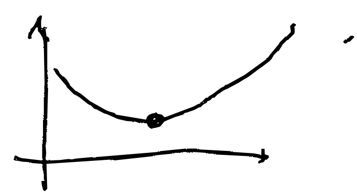
($\Phi \rightsquigarrow \Phi^{(\mu)}$ with $\Phi^{(\mu)}(x) = \Phi(\mu x)$ scalar,

so $\nabla_{A^{(\mu)}} \Phi^{(\mu)}(x) = \mu \nabla_A \Phi(\mu x)$) shows,

writing $YM(A, \Phi) = E_4 + E_2 + E_0$

$$YM(A^{(\mu)}, \Phi^{(\mu)}) = \mu^{4-d} E_4 + \mu^{2-d} E_2 + \mu^{-d} E_0$$

$\Rightarrow$ we can have interesting critical points only when $d = 2, 3$.



(Derrick's argument).

Cases of special topological interest:

•) $d=3$, $G=SO(3)$ monopoles ('t Hooft - Polyakov)
$\hookrightarrow E \to M^3$ $rk_{\mathbb{R}}=3$ orthogonal bundle.

•) $d=2$ $G=U(1)$ vortices (Abrikosov).

Let's look at monopoles in $\mathbb{R}^3$.

In this case, it's natural to ask $|\Phi| \to 1$
sufficiently fast.

Again, critical points satisfy 2nd order PDE;

Special properties as instantons in $\mathbb{R}^4$

arise setting $\lambda=0$, with

$$YMH(A, \Phi) = \frac{1}{2}\|F_A\|^2_{L^2} + \frac{1}{2}\|\nabla_A \Phi\|^2_{L^2}$$

with boundary condition $|\Phi| \to 1$

$\Phi(x) \in so(3)$, 3D, so we have

$\Phi/|\Phi| : S^2_R \to S^2 \Rightarrow$ well-defined $\deg_\infty(\Phi)$.

In this situation, we have then the Bogomolnyi bound

$$YMH(A, \Phi) = \frac{1}{2} \int |F_A - * D_A \Phi|^2 + 4\pi \deg_\infty(\Phi)$$

$\Rightarrow$ in each topological class the bound is realized by the Bogomolnyi eqns

$$\boxed{F_A - * D_A \Phi = 0} \qquad (\in \Omega^2(so(3))).$$

Solutions are called <u>Bogomolnyi monopoles</u>

<u>Why monopoles?</u> In the $U(1)$-case, they are simply $B = \operatorname{grad} \Phi$, with

$\Phi(x) = \text{const} \cdot \frac{1}{|x|}$ is the Dirac monopole!

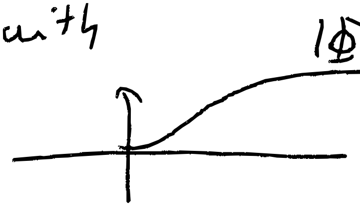
The non-abelian case has interesting smooth finite energy solutions!

<u>BPS monopole 1975</u>: found imposing equivariance. Set:

•) $\hat{r} = \frac{x}{|x|}$ unit radial vector with entries $\frac{x_i}{|x|}$

•) $so(3) \cong \mathbb{R}^3$, $\{e_j\}$ ON basis $\left(e_j = \frac{i}{\sqrt{2}}\sigma_j\right)$.

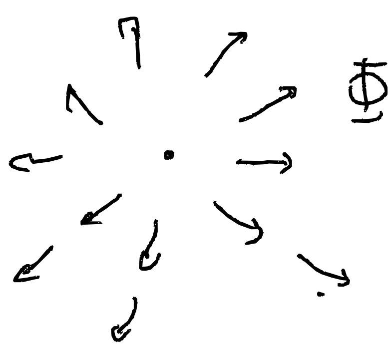
Then solution$^{(*)}$ is (A, Φ) with



$$\Phi(x) = \left(\frac{1}{\tanh r} - \frac{1}{r}\right)\hat{r}\cdot\vec{e}$$

$$A(x) = \left(\frac{1}{\sinh r} - \frac{1}{r}\right)(\hat{r}\times\vec{e})\cdot d\vec{x}$$

These are smooth at $\vec{0}$, and Φ is radial with only $\vec{0}$ at $\vec{0}$.



(*) probably off by factors of 2 ⚠

Finally, any section ϕ has a parallel part defined as $\langle \phi, \Phi \rangle_{so(3)} \frac{\Phi}{|\Phi|^2}$.

outside the zero's. (and transverse).

One computes that the parallel part of $*F_A = \nabla_A \Phi$ (1-form with values in $so(3)$) is

$$\left(\frac{1}{r^2} - \frac{1}{\sinh^2 r}\right)(\hat{r}\cdot\vec{e})\cdot\hat{r}\cdot d\vec{x},$$

asymptotic to Dirac monopole $\frac{\hat{r}}{r^2}$.

The story for vortices at $\lambda = 1$ in $\mathbb{R}^2$ is analogous, top bound based on $\deg_\infty \in \pi_1(S^1) = \mathbb{Z}$. We'll study them in detail on a surface (Σ, g)!

Lecture 5 Elliptic operators & Sobolev spaces

<u>Motivation</u> need the right functional setup to solve our equations of interest. Overall strategy: prove solution in generalized sense exist in larger space (measures, distributions, etc..), then show it's actually a nice smooth function.

$E, F \to M^n$ bundles.

<u>Def</u> A <u>differential operator</u> of order $\ell \in \mathbb{N}$ is $T : \Gamma(E) \to \Gamma(F)$ linear map that in local trivializations of E, F, local coordinates $U \subseteq \mathbb{R}^n$ on M looks like

$$\mathcal{C}^\infty(U; \mathbb{R}^{\dim E}) \to \mathcal{C}^\infty(U; \mathbb{R}^{\dim F})$$

$T = \sum_{|\alpha| \le \ell} a_\alpha(x) D^\alpha$ where:

- if $\alpha = (\alpha_1, \dots \alpha_n)$, $D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$

- $a_\alpha(x)$ is a $\dim F \times \dim E$ matrix, smooth in x.

Ex: • $\Delta = -\frac{\partial^2}{\partial x_1^2} - \cdots \frac{\partial^2}{\partial x_n^2}$ (Laplacian)

- $\frac{\partial^2}{\partial t^2} + \Delta$ (wave op), $\frac{\partial}{\partial t} + \Delta$ (heat op)

- $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right) : \mathcal{C}^\infty(\underset{\mathbb{C}}{U \cap} ; \mathbb{C}) \circlearrowleft$

Def T is elliptic if substituting

$$D_\alpha \rightsquigarrow \xi_1^{\alpha_1} \cdots \xi_n^{\alpha}, \quad \xi \in \mathbb{R}^n$$

$$P_{T,x}(\xi) = \sum_{|\alpha|=\ell} a_\alpha(x)\xi^\alpha : \mathbb{R}^{\dim E} \to \mathbb{R}^{\dim F}$$

is <u>isomorphism</u> $\forall x$, $\xi \in \mathbb{R}^n \setminus \{0\}$.

<u>Ex</u> (with constant coefficients) $(:\mathbb{R} \to \mathbb{R})$

- $\Delta \rightsquigarrow -\xi_1^2 - \dots - \xi_n^2 = -|\xi|^2$ elliptic

- $\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \rightsquigarrow \xi_1^2 - \xi_2^2$ <u>not</u> elliptic

- $\frac{\partial}{\partial x_1} - \frac{\partial^2}{\partial x_2^2} \rightsquigarrow -\xi_2^2$ <u>not</u> elliptic

- $\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} \rightsquigarrow \xi_1 + i\xi_2$ elliptic

<u>Intrinsically</u>, say $\dim E = \dim F$.

Then T of order ℓ is elliptic

$\Leftrightarrow$ $\forall x \in M$, $u \in \Gamma(E)$ s.t. $u(x) \neq 0$

$\varphi \in \mathcal{C}^\infty(M)$ s.t. $\varphi(x) = 0$ and

$d\varphi = \xi \neq 0 \in T_x^* M$,

$$T(\varphi^\ell u)(x) \neq 0 \quad \Big(= P_{T,x}(\xi)(\underset{\substack{\downarrow \\ E_x}}{u(x)}) \in F_x\Big)$$

up to scaling.

<u>Key feature</u> if T is elliptic, solutions to $Tu = 0$ are very nice.

<u>Ex</u> if $u : \Omega \to \mathbb{C}$ is $\mathcal{C}^1$ and $\frac{\partial}{\partial \bar z} u = 0$
$\Omega \underset{\sim}{\subset} \mathbb{C}$

(i.e. u is holomorphic) $\Rightarrow u \in \mathcal{C}^\infty$.

To discuss this in the generality needed in gauge theory, the right functional setup is that of <u>Sobolev spaces</u>.

$E \to M$, M Riemannian, fix ∇ on E hermitian.

$u \in \Gamma(E) \rightsquigarrow \nabla u \in \Gamma(T^*M \otimes E) \rightsquigarrow \nabla^2 u \in \Gamma(T^*M^{\otimes 2} \otimes E)$

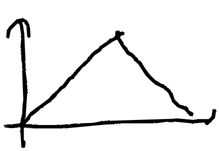
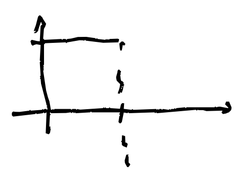
The L^2_k Sobolev norm is

$$\|u\|_{L^2_k} := \int_M |u|^2 + |\nabla u|^2 + \dots |\nabla^k u|^2 \, d\mathrm{vol}_g .$$

(induced by inner product). If <u>M cpt</u>, different (g, ∇) induce equivalent norms

<u>**Def**</u> $L^2_k(E) =$ completion of $(\Gamma(E), \|\cdot\|_{L^2_k})$

Sobolev space (Hilbert).

<u>**Ex**</u>  has differential 

$u \in L^2_1(\mathbb{R})$

$du \in L^2(\mathbb{R})$

(can define L^2_k in terms of distributions).

<u>For intuition</u>, think of functions on $\mathbb{T}^n = (\mathbb{R}/2\pi\mathbb{Z})^n$. Fourier series

$$\mathcal{F} : L^2(\mathbb{T}^n) \to \ell^2(\mathbb{Z}^n)$$ → square summable

$$f \mapsto \{\hat{f}(\underline{n})\}, \quad \hat{f}(\underline{n}) = \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f(\underline{x}) e^{-i\underline{x}\,\underline{n}} \, d\underline{x}.$$

↳ Fourier coefficients

is isometry $\|f\|_{L^2}^2 = \sum_{\underline{n}\in\mathbb{Z}^n} |\hat{f}(\underline{n})|^2$ of Hilbert spaces.

If $f \in C^1$, $\frac{d}{dx_j} f \mapsto \{ i n_j \hat{f}(\underline{n}) \}$

$$\Rightarrow \|Df\|_{L^2}^2 = \sum_{\underline{n}\in\mathbb{Z}^n} |\underline{n}|^2 |\hat{f}(\underline{n})|^2 .$$

So $\|f\|_{L^2_k}^2 \simeq \sum_{\underline{n}\in\mathbb{Z}^n} |\hat{f}(\underline{n})|^2 (1+|\underline{n}|^2)^k$.

⇝ analysis on $L^2_k(\mathbb{T}^n)$ reduces to summability of series.

Of course, if T is diff of of order ℓ, $u \in L^2_{k+\ell}(E) \Rightarrow Tu \in L^2_k(F)$. Indeed $\exists\, c = c(T, g, D, k)$ s.t

$$\|Tu\|_{L^2_k} \le c\|u\|_{L^2_{k+\ell}} \quad \forall u.$$

We have the key estimate.

<u>Thm A</u> (Gårding inequality) If T is <u>elliptic</u>, $\exists\, C$ s.t

$$\|u\|_{L^2_{k+\ell}} \le C\left(\|Tu\|_{L^2_k} + \|u\|_{L^2}\right).$$

<u>Rmk</u>) False for C^k norms!!!

•) $\|u\|_{L^2}$ is only to deal with $\ker T$; if $\ker(T) = \{0\}$, or $u \perp \ker T$, we have.

$$\|u\|_{L^2_{k+\ell}} \le C\|Tu\|_{L^2_k}.$$

<u>Intuition</u> Δ on $\mathbb{T}^2$, then

$$\Delta f \xrightarrow{\mathcal{F}} \{-|\underline{n}|^2 \hat{f}(\underline{n})\} \in \ell^2(\mathbb{Z}^n).$$

$$\Rightarrow \|\Delta f\|^2_{L^2_k} \simeq \sum_{\underline{n}\in\mathbb{Z}^n} |n|^4 |\hat{f}(\underline{n})|^2 (1+|n|^2)^k.$$

But up to a constant (k fixed)

$$|\underline{n}|^2 |\hat{f}(\underline{n})|^2 (1+|\underline{n}|^2)^k \gtrsim (1+|\underline{n}|^2)^{k+2}$$

<u>unless</u> $\underline{n}=\underline{0}$; add $|\hat{f}(0)|^2$.

($\ker\Delta = \mathbb{C} \leftrightarrow$ constant functions).

To go back to C^k norms:

<u>Thm B</u> (Sobolev embedding thm).

For $r > \frac{\dim M}{2}$,

$$L^2_{k+r} \hookrightarrow C^k \quad \text{(continuous embedding)}$$

i.e $\|u\|_{C^k} \lesssim \|u\|_{L^2_{k+r}}$.

Rmk This is sharp! In dim 2, $L^2_1 \not\hookrightarrow C^0$. ($k=0$)

Ex: On $B_{1/2} \subseteq \mathbb{R}^2$, $\log\log\left(\frac{1}{|x|}\right)$ is in L^2_1

but not continuous. But $L^2_2 \hookrightarrow C^0$.

Intuition on $\mathbb{T}^n$, $k=0$. Choose f smooth

so $f(x) = \sum_{\underline{n}\in\mathbb{Z}^n} \hat{f}(\underline{n}) e^{-i\underline{n}\cdot x}$. Then

$$|f(0)| = \Big|\sum_{\underline{n}\in\mathbb{Z}^n} \hat{f}(\underline{n})\Big| \leq \sum_{\underline{n}\in\mathbb{Z}^n} |\hat{f}(\underline{n})| =$$

$$= \sum |\hat{f}(\underline{n})| \cdot \frac{(1+|\underline{n}|^2)^{r/2}}{(1+|\underline{n}|^2)^{r/2}} \overset{\text{Cauchy-Schwartz}}{\leq}$$

$$\leq \left(\sum |\hat{f}(\underline{n})|^2 (1+|\underline{n}|^2)^r\right)^{\frac{1}{2}} \cdot \underbrace{\left(\sum \frac{1}{(1+|\underline{n}|^2)^r}\right)^{\frac{1}{2}}}_{<\infty \text{ if } r>\frac{n}{2}} \simeq \|f\|_{L^2_r}$$

For existence results:

<u>Thm C</u> (Rellich) For $k_1 > k_2$, the inclusion

$$L^2_{k_1} \hookrightarrow L^2_{k_2} \text{ is } \underline{\text{compact}}$$

(i.e a bounded sequence in $L^2_{k_1}$ has a subsequence that converges in $L^2_{k_2}$)

<u>Rmk</u> this is the Sobolev analogue of Ascoli-Arzelá ($C^{k_1} \hookrightarrow C^{k_2}$ cpt).

Prove it on T^n (exercise!).

Finally, regarding generalized solutions.

Say $u \in C^\infty(E)$, $\varphi \in C^\infty(F)$.

Then $\langle Tu, \varphi \rangle_{L^2(F)} = \langle u, T^*\varphi \rangle_{L^2(E)}$

where $T^*: \mathcal{C}^\infty(F) \to \mathcal{C}^\infty(E)$ is the (L^2 formal) adjoint diff operator.

If $Tu = v$ $(\in \mathcal{C}^\infty(F))$, then

$$\langle u, T^*\varphi \rangle_{L^2} = \langle v, \varphi \rangle_{L^2} \quad \forall \varphi \in \mathcal{C}^\infty(F) \qquad (*)$$

In this expression we are not taking derivatives of u and v!!

<u>Def</u> $u, v \in L^2$. Then we say u solves $Tu = v$ <u>weakly</u> if $(*)$ holds.

<u>Thm D</u> (Elliptic regularity) T elliptic op of order l. If $v \in L^2_k$, and $u \in L^2$ solves $Tu = v$ weakly $\Rightarrow u \in L^2_{k+l}$.

So in particular, if $u \in L^2$ solves

$Tu = 0$ weakly $\Rightarrow u \in L^2_{k+l}$ $\forall k$

$\Rightarrow u \in C^\infty$ by Sobolev embedding thm!

<u>Rmk</u> One often needs more generally

the Banach spaces L^p_k in applications,

but we'll need them minimally.

The proofs are much more involved,

especially Thm A & Thm D (which hold

for $1 < p < \infty$) & need tools such as

singular integrals, Calderon-Zygmund dec,

etc..

$\boxed{\text{Lecture 6}}$ Hodge theorem & index theory.

Let's start by finishing the proof of:

<u>Hodge theorem</u> (M,g) cpt Riemannian. Then

$$\Omega^p = \underbrace{\mathcal{H}^p}_{\text{harmonic } p\text{-forms} = \ker\Delta} \oplus \operatorname{Im}\Delta \quad (L^2\text{-orthogonal dec}).$$

$$(\Rightarrow \Omega^p = d\Omega^{p-1} \oplus \mathcal{H}^p \oplus d^*\Omega^{p+1}).$$

<u>Prop</u> $d+d^* : \Omega^*(M) \circlearrowleft$ is a first order elliptic op ($\Rightarrow \Delta = (d+d^*)^2$ is second order elliptic).

<u>Proof</u> Intrinsic def: $x\in M$, $\xi = d\varphi(x) \in T_x^*M\setminus\{0\}$ with $\varphi(x)=0$, $\alpha\in\Omega^p$. Want to show

$$(d+d^*)(\varphi\alpha)(x) \neq 0.$$

To see this, compute:

$$d(\varphi\alpha)(x) = (d\varphi\wedge\alpha + \varphi\, d\alpha)(x) = \xi\wedge\alpha.$$

$$-{*}d{*}(\varphi\alpha)(x) = {*}d(\varphi\cdot{*}\alpha)(x) = \dots = {*}(\xi\wedge{*}\alpha)$$

$\Rightarrow d^*(\varphi\alpha)(x) = \xi \lrcorner\, \alpha$ (where $\lrcorner$

is $e\lrcorner\,\omega := (-1)^{n(p+1)+1}{*}(e\wedge{*}\omega)$, usual contraction

with $e^\flat$ up to sign).

$\Rightarrow$ symbol is $\xi\wedge\alpha + \xi\lrcorner\,\alpha \neq 0$.

<u>Proof of Hodge thm</u> write $\Omega^p = \mathcal{H}^p \oplus (\mathcal{H}^p)^\perp$,

want to prove $(\mathcal{H}^p)^\perp = \Delta(\Omega^p)$.

$\supseteq$) is clear: if $\alpha = \Delta\beta$, $\omega\in\mathcal{H}^p$

$\Rightarrow \langle\alpha,\omega\rangle_{L^2} = \langle\Delta\beta,\omega\rangle_{L^2} = \langle\beta,\Delta\omega\rangle_{L^2} = 0.$

Non trivial part is $\subseteq$!

Rmk we don't know yet $\mathcal{H}^p \cong H^p(M;\mathbb{R})$!

If $\alpha \in (\mathcal{H}^p)^\perp$, want to show $\exists\, \beta \in \Omega^p$ s.t. $\alpha = \Delta\beta$. In fact, by elliptic regularity, we only need to find a weak solution, i.e $\beta \in L^2(\Omega^p)$ s.t.

$$\langle \alpha, \varphi \rangle_{L^2} = \langle \beta, \Delta\varphi \rangle_{L^2} \quad \forall \varphi \in \Omega^p.$$

But β exists by abstract nonsense as follows.

Define $\ell : \operatorname{Im}\underset{\subseteq \Omega^p}{\Delta} \to \mathbb{R}$

$$\Delta\varphi \mapsto \langle \alpha, \varphi \rangle_{L^2}$$

•) well-defined: if $\Delta\varphi = \Delta\varphi' \to \varphi - \varphi' \in \mathcal{H}^p$

$$\langle \alpha, \varphi \rangle_{L^2} = \langle \alpha, \varphi' \rangle_{L^2}.$$

•) <u>bounded</u>, because if

$$\varphi = \varphi^{\mathrm{harm}} + \varphi^{\perp} \in \mathcal{H}^p \oplus (\mathcal{H}^p)^{\perp},$$

$$|\ell(\Delta\varphi)| = |\ell(\Delta\varphi^{\perp})| = |\langle \alpha, \varphi^{\perp}\rangle_{L^2}|$$

$\leq \|\alpha\|_{L^2}\|\varphi^{\perp}\|_{L^2}$ (by Cauchy-Schwarz)

$\leq \|\alpha\|_{L^2}\|\varphi^{\perp}\|_{L^2_2}$

$\leq C\|\alpha\|_{L^2}\|\Delta\varphi^{\perp}\|_{L^2}$ (Gårding inequality).

$= C\|\alpha\|_{L^2}\|\Delta\varphi\|_{L^2}$.

By Hahn-Banach, ℓ extends to bounded

$\ell : L^2(\Omega^p) \longrightarrow \mathbb{R}$. (i.e. $\ell \in L^2(\Omega^p)^*$).

↳ Hilbert space

By Riesz representation, $\ell(-) = \langle \beta, - \rangle_{L^2}$

for $\beta \in L^2(\Omega^p)$, which is weak solution

of $\Delta\beta = \alpha$ $\square$.

Elliptic operators have very interesting topological properties!

<u>Basic observation</u> $T: \mathbb{R}^n \to \mathbb{R}^m$ linear map.

By rank-nullity,

$$\operatorname{ind}(T) := \underbrace{\dim \ker T - \dim \operatorname{coker} T}_{\text{depends on } T} = \underbrace{n - m}_{\text{independent of } T}$$

$\left(= \mathbb{R}^m / \operatorname{Im} T\right)$

<u>Def</u> X, Y Hilbert spaces (or Banach),
$T: X \to Y$ bounded op is <u>Fredholm</u> if

a) $\dim \ker T < \infty$

b) $\dim \operatorname{coker} T < \infty$

c) $\operatorname{Im} T$ closed.

The <u>index</u> of T is

$$\operatorname{ind}(T) := \dim \ker T - \dim \operatorname{coker} T \in \mathbb{Z}.$$

Rmk in this setup, 2) $\Rightarrow$ 3) by open mapping thm.

Ex •) $R : \ell^2(\mathbb{N}) \circlearrowleft$ right shift

$(a_0, a_1, \dots) \mapsto (0, a_0, a_1, \dots)$

has $\operatorname{ind} R = 0 - 1 = -1$.

•) $L : \ell^2(\mathbb{N}) \circlearrowleft$ left shift

$(a_0, a_1, \dots) \mapsto (a_1, \dots)$

has $\operatorname{ind} L = 1 - 0 = 1$

Thm T order ℓ elliptic op on M cpt, then

$\forall k \quad T : L^2_{k+\ell} \to L^2_k$ is Fredholm,

and $\operatorname{ind}(T)$ is independent of k

(in fact, $\operatorname{coker} T \cong \ker T^*$ $\leftarrow$ formally adjoint diff op)

Let's prove $\dim \ker L$ is finite.

Abstract Lemma $X \xrightarrow{L} Y$, $X \xrightarrow{k} Z$ s.t

1) k is cpt, 2) $\|x\|_X \leq C(\|Lx\|_Y + \|kx\|_Z)$

$\Rightarrow \ker L$ is finite dimensional.

Proof If $\dim \ker L = \infty$, $\exists \{x_i\} \subseteq \ker L$

s.t $\|x_i\|_X \leq 1$, $\|x_i - x_j\|_X \geq c$. Up to subsequence,

$\{kx_i\}$ converges, but 2) implies.

$\|kx_i - kx_j\|_Z \gtrsim \|x_i - x_j\|_X \geq 1$, contradiction $\square$.

Apply lemma to

$L^2_{k+1} \xrightarrow{T} L^2_k$, $L^2_{k+1} \hookrightarrow L^2$ 1) is Rellich's thm 2) is Gårding ineq.

$\underline{\text{Thm}}$ not hard!

$\mathrm{Fred}(X,Y) \subseteq B(X,Y)$ is open,

Fredholm bounded

and $\mathrm{ind} : \mathrm{Fred}(X,Y) \to \mathbb{Z}$ is locally constant.

$\underline{\text{Cor}}$ $T_t : \Gamma(E) \to \Gamma(F)$ 1-parameter family of elliptic ops on M cpt $\Rightarrow \mathrm{ind}(T_t) \in \mathbb{Z}$ is constant.

$\underline{\text{Ex}}$ if T is self adjoint (e.g. $d+d^* : \Omega^* \to \Omega^*$)

$\mathrm{coker}\, T = \ker T^* = \ker T \Rightarrow \mathrm{ind}\, T = 0$.

More interestingly, consider

$D = d + d^* : \Omega^{\text{even}} \longrightarrow \Omega^{\text{odd}}$. (elliptic!)

$\Rightarrow \ker = \bigoplus \mathcal{H}^{\text{even}} = \bigoplus H^{\text{even}}(M;\mathbb{R})$

$\mathrm{coker} = \ker(d+d^* : \Omega^{\text{odd}} \to \Omega^{\text{even}})$

$= \bigoplus H^{\text{odd}}(M;\mathbb{R})$

$\Rightarrow \operatorname{ind}(D) = \chi(M)$

Now, choose $\alpha \in \Omega^{odd}(M)$. Then

$D + t(\alpha \wedge -) : \Omega^{even} \to \Omega^{odd}$ is

1-parameter family of elliptic ops

$\Rightarrow \operatorname{ind}(D + (\alpha \wedge -)) = \chi(M)$!

So if $\chi(M) > 0$ (e.g. $M = S^{24}$)

$\ker(D + (\alpha \wedge -)) \neq 0$

$\rightsquigarrow$ topology forces solvability of elliptic PDEs.

Atiyah-Singer index thm (1963). For

$T : \Gamma(E) \to \Gamma(F)$ elliptic on M cpt,

purely topological (e.g. char classes of TM, E, F)

explicit formula for $\operatorname{ind}(T) \in \mathbb{Z}$.

Back to gauge theory

The (linear part) of the equations we study.

is never elliptic because of gauge invariance.

Consider for example

$$F_A^+ = 0 \in \Omega^2(\mathfrak{g}_E) \quad \text{ASD (or instanton) eqn.}$$

Locally, if $\nabla_A \stackrel{\text{loc}}{=} d + \tilde{A}$ $\quad \tilde{A} \in \Omega^1(\mathfrak{g})$

$$F_A^+ = (d\tilde{A} + \tilde{A} \wedge \tilde{A})^+$$

So the highest order term is

$$\begin{tikzcd} \Omega^1(\mathfrak{g}) \arrow[r, "d"] \arrow[rr, bend right, "d^+"'] & \Omega^2(\mathfrak{g}) \arrow[r, "\pi^+"] & \Omega^+(\mathfrak{g}) \end{tikzcd}$$

and of course $d^+(\underbrace{df}_{\text{or anything closed}}) = 0$. $\Rightarrow$ not elliptic.

(comes from $F^+_{u \cdot A} = 0 \quad \forall u \in \mathcal{G}$).

This issue is resolved by suitably <u>gauge fixing</u>. Let's do the $U(1)$ case

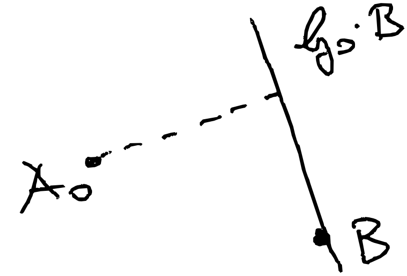
(non-abelian case <u>highly</u> non-trivial,

cf <u>Uhlenbeck's gauge fixing thm</u> (1982))

$L \to (M,g)$ cpt, fix base connection A_0.

$\mathcal{G}_0 = \{u = e^f, f \in i\Omega^0\}$ id. component of $\mathcal{G}$.

Then $e^f \cdot B = A - df$



It's natural to consider

$\tilde{B}$ that minimizes $\|u \cdot B - A_0\|_{L^2}$.

This solves (usual computation)

$$\boxed{d^*(B - A_0) = 0}$$ <u>Coulomb gauge condition</u>.

(12)

Hodge theorem $\Rightarrow \forall B, \exists! f$ up to overall constant s.t $e^f \cdot B$ is is Coulomb gauge.

Now, the first order part of ASD eqn + Coulomb gauge fixing is

$$d^* + d^+ : i\Omega^1(M) \to i\Omega^0(M) \oplus i\Omega^+(M)$$

which is elliptic! (dimensions check: $4 = 1+3$)

One also checks $\ker = \mathcal{H}^1$, and

$$\operatorname{coker} = \underset{\text{constants}}{\mathbb{R}} \oplus \mathcal{H}^+,$$ self-dual harmonic 2-forms,

$(+1)$ eigenspace of $* : \mathcal{H}^2 \circlearrowleft$, whose dim is the $b^+(X)$ = positivity of quadratic form

$$Q_X : H^2(X;\mathbb{R}) \to \mathbb{R} \quad \alpha \mapsto (\alpha \cup \alpha)[X].$$

$\Rightarrow \text{ind} = b_1 - b^+ - 1.$ (top quantity!)

Lecture 7 The vortex equations on a surface

<u>Geometric setup</u> (Σ, g) oriented Riemannian surface,

$L \to \Sigma$ hermitian line bundle.

$\mathcal{C} = \{(A, \Phi) \mid A$ compatible connection on L, $\Phi \in \Gamma(L)\}$

$\mathcal{G} = \{u : \Sigma \to U(1)\} \curvearrowright \mathcal{C}$ via

$$u \cdot (A, \Phi) = (A - u^{-1} du, u \cdot \Phi)$$

We consider the <u>Yang-Mills-Higgs functional</u>

$$YMH(A, \Phi) = \frac{1}{2}\int |F_A|^2 + \frac{1}{2}\int |\nabla_A \Phi|^2 + \frac{\lambda}{8}\int (1 - |\Phi|^2)^2,$$

$YMH : \mathcal{C}/\mathcal{G} \to \mathbb{R}^{\geq 0}$. (fixed parameter $\lambda > 0$)

<u>Goal</u> topological bounds in the case of

<u>critical coupling</u> $\lambda = 1$; ($\lambda < 1$, $\lambda > 1$

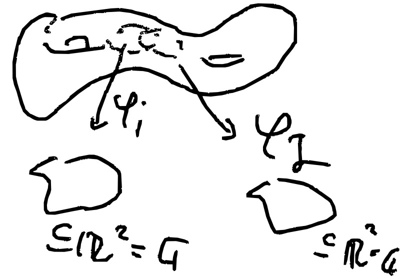
describe Type I vs Type II superconductors)

The natural top. invariant here is $c_1(L) \in H^2(\Sigma;\mathbb{Z}) \cong \mathbb{Z}$, also called the degree $d = d(L) \in \mathbb{Z}$. Facts:

- it's the only top invariant of line bundles on Σ: if $d(L) = d(L') \Rightarrow L \cong L'$.
- it's equal to the Euler number of $L_{\mathbb{R}}$, i.e the # zeroes of a generic section $L \underset{s}{\rightleftarrows} \Sigma$ (counted with signs).

The bounds are based on the interplay between the Riemannian and complex geometry of surfaces (simplest $\dim_{\mathbb{C}} = 1$ case of Kähler geometry), which we now discuss.

A <u>Riemann surface</u> S (= complex mfld of dim 1) is a surface Σ with given atlas for which the transition maps are



<u>biholomorphisms</u>

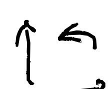
(= orientation preserving conformal diffeomorphism).

$\Rightarrow$ well-defined notion of $f : \Sigma \to \mathbb{C}$ <u>holomorphic</u>

These charts endow Σ with an almost complex structure

$$\begin{array}{ccc} T\Sigma & \xrightarrow{J} & T\Sigma \\ & \searrow \ \Sigma \ \swarrow & \end{array},$$

bundle map s.t. $J^2 = -\mathrm{Id}$ induced by counter clockwise 90° rotation in $\mathbb{C}$

$\left(\frac{\partial}{\partial x} \mapsto \frac{\partial}{\partial y},\ \frac{\partial}{\partial y} \mapsto -\frac{\partial}{\partial x} \text{ in } \mathbb{R}^2 \right.$  $\left. \right)$

$$S \xrightarrow{\text{forget}} (\Sigma, J)$$

In this language, $f : \Sigma \to \mathbb{C}$ is holomorphic

$\Leftrightarrow$ $df : T\Sigma \to \mathbb{C}$ is $\mathbb{C}$-linear at all points

$\boxed{i \circ df = df \circ J}$ Cauchy–Riemann equations.

(If $f = u + iv$ in $\mathbb{C}$, this is just $\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$)

Notice that an oriented Riemannian surface

(Σ, g) also induces $J : TM^{\circ}$ a.c.s.

$$(\Sigma, g) \xrightarrow{\text{forget}} (\Sigma, J).$$

Of course, every (Σ, J) is induced by

(Σ, g) (and any other is $g' = e^{2f} g$.)

(this is just linear algebra!).

More subtle is

<u>Thm</u> (1D Newlander–Nirenberg) Every (Σ, J)

is induced by (a unique) Riemann surface

structure.

This is an <u>integrability</u> result;

need to show $\forall x \in \Sigma$, $\exists U \ni x$ and

$f : U \to \mathbb{C}$ diffeo onto image satisfying

CR equations. ($\Rightarrow f$ gives local chart).

<u>Idea of proof</u> (we'll prove a related

integrability result next time).

exploit holomorphic vs harmonic.

Fix g inducing J. Then $* : \Omega^1(\Sigma) \circlearrowleft$

(sends $dx \mapsto dy$, $dy \mapsto -dx$ in $\mathbb{R}^2$)

(so $* = -J^* \to$ dual action).

key point!

<u>Find</u> by PDE techniques $u : \hat{V}^x \to \mathbb{R}$

s.t $\Delta u = 0$ and $du_x \neq 0$.

$\hookrightarrow$ uses g!

Then $d(* du) = 0 \Rightarrow * du$ closed

$\Rightarrow \exists\, v$ (unique up to constant) s.t

$* du = dv$, and $u+iv$ solves CR equations

(and is local diffeo) $\Box$.

From now on, we'll always think $(\Sigma, g) \rightsquigarrow S$.

(i.e Riemannian surface $\Rightarrow$ Riemann surface).

A useful way to rephrase CR eqns is

via (p,q)-forms:

$J^* \curvearrowright T^*_x \Sigma$ with $(J^*)^2 = -I$

$\Rightarrow \Omega^1 \otimes \mathbb{C} := \Omega^{1,0} \oplus \Omega^{0,1}$ $(\pm i)$-eigenspaces

in $\mathbb{R}^2$: $\Omega^{1,0} = \operatorname{Span}(dz = dx + i dy)$, $\Omega^{0,1} = \operatorname{Span}(d\bar{z} = dx - i dy)$.

Rmk we also have $\Omega^2 \otimes \mathbb{C} = \Omega^{1,1}$ ($=$ span $dz\, d\bar{z}$)

We decompose

$$d = \partial + \bar{\partial} : \Omega^0 \otimes \mathbb{C} \xrightarrow{d} \Omega^1 \otimes \mathbb{C} = \Omega^{1,0} \oplus \Omega^{0,1},$$

and f is holomorphic $\Leftrightarrow$ $\bar{\partial} f = 0$.

<u>Rmk</u> by maximum principle, $f : \underset{\text{cpt}}{\Sigma} \to \mathbb{C}$ hol is constant.

<u>Back to gauge theory</u>

Suppose now we have on (Σ, g) a hermitian line bundle L ($\cong \mathbb{C}$ locally),

$$\Rightarrow \Omega^1(L) = \Omega^{1,0}(L) \oplus \Omega^{0,1}(L).$$

If A is compatible connection, we have

$$d_A := \partial_A + \bar{\partial}_A : \Omega^0(L) \to \Omega^1(L) = \Omega^{1,0}(L) \oplus \Omega^{0,1}(L).$$

$\partial_A, \bar{\partial}_A$ 1-st order elliptic operators;

we also have the version involving $2 = (1,1)$ forms.

$$\begin{tikzcd}
 & \Omega^{1,1}(L) & \\
\Omega^{1,0}(L) \arrow[ur, "\bar\partial_A"] & & \Omega^{0,1}(L) \arrow[ul, "\partial_A"'] \\
 & \Omega^{0}(L) \arrow[ul, "\partial_A"] \arrow[ur, "\bar\partial_A"'] &
\end{tikzcd}
\qquad \updownarrow *$$

$\longleftrightarrow$
conjugation

All of these are inner product spaces; the adjoints can be explicitly written down, e.g. the adjoint of $\bar\partial_A : \Omega^0(L) \to \Omega^{0,1}(L)$ is $\bar\partial_A^* = -\bar{*}\,\bar\partial_A\,\bar{*}$ $\Omega^{1,0}(L) \to \Omega^{1,1}(L)$.

We also record:

<u>Prop</u> (1D Kähler identities) The adjoint of

$\partial_A : \Omega^0(L) \to \Omega^{1,0}(L)$ $\left(\bar\partial_A : \Omega^0(L) \to \Omega^{0,1}(L)\right)$

is $i * \bar\partial_A$ $\quad (-i * \partial_A)$

Thm (top bound) when $\lambda = 1$, we have

$$YMH(A,\Phi) = \|\underbrace{\bar\partial_A \Phi}_{\Omega^{0,1}(L)}\|^2_{L^2} + \frac{1}{2}\|\overbrace{*F_A}^{\in i\Omega^0} - \frac{i}{2}(|\Phi|^2 - 1)\|^2_{L^2} + \pi \deg(L).$$

Proof If we expand the second term, we obtain the terms:

- $\frac{1}{2}\|F_A\|^2_{L^2}$, $\frac{1}{8}\|1-|\Phi|^2\|^2_{L^2}$ which appear in YMH ($\lambda = 1$!!)
- $\int_\Sigma \frac{i}{2} F_A = \pi c_1(L)$ by Chern-Weil
- $\frac{1}{2}\langle i*F_A, |\Phi|^2\rangle_{L^2}$

But $\langle i*F_A, |\Phi|^2\rangle_{L^2} = \langle i*F_A\Phi, \Phi\rangle_{L^2}$

$= \langle i*d_A d_A\Phi, \Phi\rangle_{L^2} =$ (using $d_A = \partial_A + \bar\partial_A$)

(10)

$$\langle i * \partial_A \bar\partial_A \Phi, \Phi \rangle_{L^2} + \langle i * \bar\partial_A \partial_A \Phi, \Phi \rangle_{L^2}$$

$$= -\|\bar\partial_A \Phi\|^2_{L^2} + \|\partial_A \Phi\|^2_{L^2} \quad \text{by Kähler id.}$$

We conclude by $\|\underset{d_A}{\overset{\shortparallel}{\nabla_A}} \Phi\|^2_{L^2} = \|\partial_A \Phi\|^2_{L^2} + \|\bar\partial_A \Phi\|^2_{L^2}$ $\square$.

<u>Cor</u> If $L \to (\Sigma, g)$ has $\deg L = d \geq 0$, then

$$YMH(A, \Phi) \geq \pi d \quad \text{(top bound)}$$

and bound is achieved <u>if and only if</u>

$$\begin{cases} \bar\partial_A \Phi = 0 & (\in \Omega^{0,1}(L)) \\ * F_A = \frac{i}{2}(|\Phi|^2 - 1) & (\in i\Omega^0) \end{cases}$$

<u>vortex equations</u> (1st order, non-linear gauge invariant PDEs)

We'll focus on $\Phi \not\equiv 0$; if not, it's just

$*F_A = -\frac{i}{2}$ constant (central) curvature eqn. (11)

<u>Notice</u>: if solution (A, Φ) exists,

$$\deg(L) = \int_\Sigma \frac{i}{2\pi} F_A = \frac{1}{4\pi} \int_\Sigma (1 - |\Phi|^2)\, d\mathrm{vol} < \frac{\mathrm{Vol}(\Sigma)}{4\pi}.$$

Our goal for the next few lectures is to prove

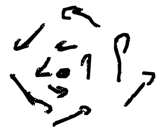
<u>Thm</u> (Taubes '80, Bradlow '89) $L \to (\Sigma, g)$ with $0 \leq \deg(L) < \frac{\mathrm{Vol}(\Sigma)}{4\pi}$. Then $\exists$

$$\underbrace{\left\{ \begin{array}{l} \text{solutions } (A,\Phi),\ \Phi \not\equiv 0 \\ \text{to vortex eqns} \end{array} \right\} \Big/ \mathcal{G}}_{\mathcal{M}} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{l} \text{unordered } d\text{-uples} \\ \text{of points } x_1, \ldots x_d \in \Sigma \end{array} \right\}$$

natural bijection (to be described).

<u>Def</u> RHS is $\Sigma^d / S_d =: \mathrm{Sym}^d(\Sigma)$ <u>symmetric product</u> of Σ.

Rmk Taubes proved the case $\Sigma = \mathbb{R}^2 = \mathbb{C}$ with condition $|\Phi| \to 1$ at ∞.

Basic vortex  (justifies name)

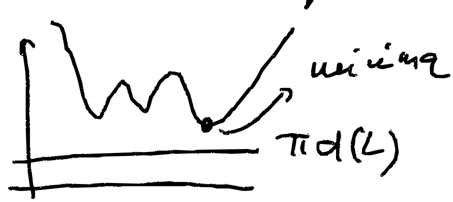
We'll follow the proof of Garcia-Prada (92) which also shows that $\mathcal{M}$ is naturally a $2d$-dimensional smooth mfld equipped with a symplectic form.

One can easily see $\mathrm{Sym}^d(\Sigma)$ is topological mfld using the charts

$$\mathbb{C}^d/S_d \longleftrightarrow \mathbb{C}^d$$

$$[z_1, \dots z_d] \longmapsto \text{coeff. of } (z-z_1)\cdots(z-z_d).$$

Rmk The "naive" variational strategies will not work YMH



$\boxed{\text{Lecture 8}}$ Geometry of holomorphic line bundles.

Σ Riemann surface, $L \to \Sigma$ $\mathbb{C}$-line bundle (no metrics here!) so locally $L|_{U_i} \xrightarrow{\tau_i} U_i \times \mathbb{C}$, transition functions $g_{ij} : U_i \cap U_j \to GL(1,\mathbb{C}) = \mathbb{C}^*$.

$$\left(\begin{tikzcd} U_i \cap U_j \times \mathbb{C} \arrow[rr, bend right, "g_{ij}"'] & L|_{U_i \cap U_j} \arrow[l, "\tau_i"', "\sim"] \arrow[r, "\tau_j", "\sim"'] & U_i \cap U_j \times \mathbb{C} \end{tikzcd} \right)$$

<u>Def</u> A <u>holomorphic line bundle</u> $\mathcal{L} \to \Sigma$ (with underlying smooth bundle L) is a collection of trivializations $\{(U_i, \tau_i)\}$ s.t the transition functions g_{ij} are holomorphic.

We have a well defined operator

$$\bar\partial_{\mathcal{L}} : \Omega^0(L) \to \Omega^{0,1}(L).$$

②

If the section $s \in \Omega^0(L)$ is given in trivializations by s_i, s_j, $s_j = g_{ij} s_i$

so $\bar\partial s_j = \cancel{(\bar\partial g_{ij})} \cdot s_i + g_{ij} \bar\partial s_i$.

↳ holomorphic

The operator $\bar\partial_L$ is the $(0,1)$ version of a covariant derivative

$$\bar\partial_L(fs) = \bar\partial f \cdot s + f \cdot \bar\partial_L s \quad \forall f \in \Omega^0_{\mathbb{C}},\ s \in \Omega^0(L)$$

((0,1)-Leibniz rule).

<u>Def</u> For $L \to S$, set

$$\mathcal{A}^{0,1} = \{ d^{(0,1)} : \Omega^0(L) \to \Omega^{0,1}(L) \text{ satisfying } (0,1)\text{-Leibniz rule} \}.$$

$\mathcal{A}^{0,1}$ is affine space / $\Omega^{0,1}$.

$\Omega^{0,1}(\mathrm{End}_{\mathbb{C}}(L,L))$

Key integrability result:

<u>Thm</u> (1D Koszul–Malgrange '58) Every $d^{(0,1)} \in \mathcal{A}^{(0,1)}$ is $\bar{\partial}_{\mathcal{L}}$ for some holomorphic structure $\mathcal{L}$ on L.

<u>Rmk</u> True for higher rank $E \to S$ too! Our proof readily adapts. (Riemann surface)

To prove this, we need to find trivialization defining a holomorphic structure!

<u>Concretely, want</u>: $\forall x \in S$, $\exists U \ni x$ nbhd & section s of $L|_U$ s.t. 1) $d^{(0,1)} s \equiv 0$

2) $s \neq 0$ everywhere on U.

Then 2) gives $L_U \cong U \times \mathbb{C}$, 1) implies transitions are holomorphic.

Now, fix σ any section on U nowhere vanishing, so that

$d^{(0,1)}\sigma = \theta\sigma$ for some $\theta \in \Omega^{0,1}$ (4)

For any $g : U \to \mathbb{C}^*$, $g\sigma$ is again nowhere vanishing, and

$$d^{(0,1)}(g\sigma) = (\bar\partial g)\sigma + g \cdot d^{(0,1)}\sigma =$$
$$= (\bar\partial g)\sigma + g\theta\sigma.$$

So $s = g\sigma$ solves the problem provided

$$\boxed{g^{-1}\bar\partial g = -\theta} \quad \in \Omega^{0,1}.$$

We'll show that we can solve this PDE in g $\forall \theta \in \Omega^{0,1}$, possibly in a smaller nbhd.

(we follow the proof of Atiyah-Bott '83)

<u>Strategy</u>: 1) linear problem on $\mathbb{P}^1$ → Riemann sphere

2) non-linear problem on $\mathbb{P}^1$

3) local non linear problem.

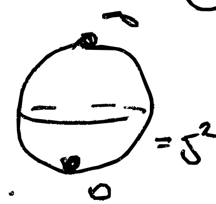
5

Here $\mathbb{P}^1$ is given by two charts

$\mathbb{C}_z = \mathbb{P}^1 \setminus \{\infty\}$, $\mathbb{C}_w = \mathbb{P}^1 \setminus \{0\}$

related by $z \mapsto \frac{1}{w}$ on $\mathbb{C}_z^* \xrightarrow{\sim} \mathbb{C}_w^*$;

we can fix the standard round metric.



$\boxed{\text{Step 1}}$ Lemma On $\underline{\mathbb{C}} \to \mathbb{P}^1$ trivial bundle, the

$\bar{\partial} : L^2_{k+1}(\Omega^0_{\mathbb{C}}) \to L^2_k(\Omega^{0,1})$ has

$\ker \cong \mathbb{C} =$ constants, $\operatorname{coker} = \{0\}$; in particular,

$\bar{\partial}|_{\langle \text{constants} \rangle^\perp}$ is isomorphism

Proof by ellipticity, can work smooth stuff.

$\ker \bar{\partial} =$ holomorphic functions $=$ constants.

$\operatorname{coker} \bar{\partial} = \ker\left(\bar{\partial}^* : \Omega^{0,1} \to \Omega^0_{\mathbb{C}}\right)$

$\qquad\qquad\qquad\; \| $

$\qquad\qquad -\bar{*}\,\bar{\partial}\,\bar{*}$ with $\bar{\partial} : \Omega^{1,0} \to \Omega^{1,1}$

So $\hat{\pi}$ identifies coker $\bar\partial$ with

$\ker(\bar\partial : \Omega^{1,0} \to \Omega^{1,1})$ = holomorphic 1-forms

(stuff locally $f(z)\,dz$, f holomorphic)

(Serre duality!). But there's no hol

1-forms on $\mathbb{P}^1$ (check using $dz = -\frac{1}{w^2}dw$
+ power series expansion) $\square$.

To deal with the non-linearity (coming from
multiplication), we use

Thm E (Sobolev multiplication thm). $k \geq j \geq 0$,
and $k > \frac{\dim M}{2}$. Then multiplication
defines continuous map

$$L^2_k \times L^2_j \to L^2_j .$$

(in particular, L^2_k is Banach algebra).

$\boxed{\text{Step 2}}$ **Prop** On $\mathbb{P}^1$, if $\theta \in \Omega^{0,1}$ has sufficiently small L^2_1-norm, then $\exists\, g : \mathbb{P}^1 \to \mathbb{C}^*$ smooth s.t. $-\theta = g^{-1}\bar\partial g$.

Proof Set $\mathcal{U}_2 = \left\{ \begin{array}{l} f \neq -1 \text{ everywhere} \\ f \perp \text{constants.} \end{array} \right\} \subseteq L^2_2(\langle\text{constants}\rangle^\perp)$

well defined open set ($L^2_2 \hookrightarrow C^0$ in dim 2),

consider the map (we'll set $g = 1+f$ later)

$$\Phi : \mathcal{U}_2 \to L^2_1(\Omega^{0,1}) \quad f \mapsto (1+f)^{-1}\bar\partial(1+f)$$

which is well-defined by Sobolev multiplication,

and in fact <u>smooth</u> map

(Here F differentiable in <u>Banach</u> setup means

$$F(x_0+h) = F(x_0) + Lh + o(\|h\|)$$

with L <u>bounded</u> operator)

with derivative at $f_0 \equiv 0$ given by

$$D_{f_0} P : L^2_2(\langle \text{constants} \rangle^\perp) \longrightarrow L^2_1(\Omega^{0,1}),$$
$$f \longmapsto \bar\partial f$$

which is iso by Lemma! Inverse function thm $\Rightarrow$ P local diffeo onto nbhd of $P(f_0) = 0$. So if $\|\theta\|_{L^2_1} < \varepsilon$, $\exists f \in \mathcal{U}_2 \subseteq L^2_2$ s.t. $(1+f)^{-1}\bar\partial(1+f) = -\theta$; set $g = 1+f \in L^2_2$.

To conclude g is smooth: $g \in L^2_2$, θ smooth
by assumption
$\Rightarrow \theta \in L^2_k \ \forall k$. Then

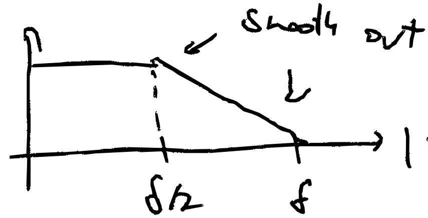
$$\bar\partial g = -g\theta \in L^2_2 \Rightarrow g \in L^2_3$$
elliptic reg

$\Rightarrow g\theta \in L^2_3 \Rightarrow g \in L^2_4 \Rightarrow \ldots$ (this is called elliptic bootstrapping) $\square$.

$\boxed{\text{Step 3}}$ To conclude local solvability of

$g^{-1}\bar{\partial} g = -\theta$, we can first assume

$\theta(0) = 0$ (change $g \rightsquigarrow g(1+c\bar{z})$ for suitable $c \in \mathbb{C}$)

Consider $\rho_\delta(|z|) =$ 

,

and let $\phi_\delta = \rho_\delta \cdot \theta \in \Omega^{0,1}(U)$, thought of as $(0,1)$-form on $\mathbb{P}^1$. Key estimate:

$\|\phi_\delta\|_{L^2_1} \to 0$ for $\delta \to 0$, so by Prop

we solve $g^{-1}\bar{\partial} g = \phi_\delta$, which is

$\equiv -\theta$ on $|z| < \delta/2$. $\square$

<u>Recap</u> we showed for $L \to S$

$$\{\text{hol structures on } L\} \xrightarrow{1:1} \{ d^{(0,1)} : \Omega^0(L) \to \Omega^{0,1}(L)\}$$

$$\mathcal{L} \longmapsto \bar{\partial}_{\mathcal{L}} \qquad\qquad \Omega^{0,1}(L) = \mathcal{A}^{0,1}.$$

Now, <u>fix</u> hermitian metric h on L, then we also have

$$\mathcal{A} = \{\text{hermitian connections on } L\} \longrightarrow \mathcal{A}^{0,1}$$

$$A \longmapsto \bar{\partial}_A$$

where $\bar{\partial}_A$ is $(0,1)$-part of $d_A : \Omega^0(L) \to \Omega^1(L)$

<u>Prop</u> This is also a bijection! i.e. given $d^{0,1} \in \mathcal{A}^{0,1}$, $\exists!\ A \in \mathcal{A}$ s.t. $\bar{\partial}_A = d^{0,1}$.

<u>Proof</u> we write an explicit formula for A!

Fix a trivialization $L|_U \cong U \times \mathbb{C}$

$\vec{e_0} \equiv 1$ constant

given by a section s with $d^{(0,1)}s = 0$.

This does not need to preserve the hermitian structure! Write a general connection

$\nabla_A = d + A$ with $A \in \Omega^1 \otimes \mathbb{C}$.

Now, $\bar{\partial}_A = d^{0,1} \Rightarrow \bar{\partial}_A e_0 = 0 \Rightarrow A \in \Omega^{1,0}$.

Imposing hermiticity for h:

$d\, h(e_0, e_0) = h(d_A e_0, e_0) + h(e_0, d_A e_0)$,

so setting $H = h(e_0, e_0)$ $(: U \to \mathbb{R})$

$dH = HA + H\bar{A}$, taking $(1,0)$-part

$\partial H = HA \Rightarrow A = H^{-1}\partial H$.

$\square$.

Putting everything together, for $L \to S$ hermitian $\exists$ correspondence

$\{\text{hol structures}\} \xrightarrow{1:1} \{\text{hermitian connections}\}$

$$\mathcal{L} \mapsto \text{unique } A \text{ s.t. } \bar\partial_A = \bar\partial_{\mathcal{L}}$$

(this is called <u>Chern connection</u>).

<u>Useful consequence</u> a solution $\Phi \in \Gamma(L)$ to the equation $\bar\partial_A \Phi = 0$ locally looks like a holomorphic function $f : U \to \mathbb{C}$ (in suitable trivialization) $\Rightarrow$ either $\Phi \equiv 0$, or zeroes isolated with well-defined multiplicity > 0 ($f(z) = a_k z^k + a_{k+1} z^{k+1} + \dots \quad a_k \neq 0$).

by <u>unique continuation</u>; topologically,

$\#\{\text{zeroes of } \Phi\} = \deg L$.

$\boxed{\text{Lecture 9}}$ Moment maps in symplectic geometry & gauge theory

electromagnetism : $U(1)$-gauge theory

= classical mechanics : symplectic geometry.

<u>Def</u> A <u>symplectic mfld</u> (M^{2n}, ω) is a smooth mfld equipped with $\omega \in \Omega^2(M)$ s.t

1) $d\omega = 0$ (closed)

2) $\omega^n \neq 0$ everywhere (ω is non degenerate skew-symmetric)

<u>Examples</u> 1) $(\underset{\underset{(x_1, y_1, \dots x_n, y_n)}{\uparrow}}{\mathbb{R}^{2n}}, \omega_{std})$, $\omega_{std} = \sum_{j=1}^{n} dx_j \wedge dy_j$.

2) More generally, $M = T^*X$ $\rightsquigarrow$ smooth mfld

has natural symplectic form ω_{can}:

in coordinates $\{x_1, \dots x_n\}$ on X, so

$T_x^* X = \{\sum \xi_j (dx_j)_x\}$ has coordinates $\{\xi_j\}$

$\omega_{can} = \sum dx_j \wedge d\xi_j$ (well-defined!)

3) Σ surface, ω volume form $\Rightarrow (\Sigma, \omega)$ symplectic.

4) S^{2n} does not have a symplectic structure ω

if $n > 1$, because $[\omega]^n$ has to be $\neq 0$.

5) $\mathbb{C}P^n$ has a natural symplectic structure ω_{FS};

(can write down explicitly)

<u>Darboux's thm</u> locally, $(M, \omega) \cong (\mathbb{R}^{2n}, \omega_{std})$

So no local invts!

<u>Rmk</u> 2) $\Rightarrow$ true pointwise, 1) $\Rightarrow$ integrability.

Recall given $H : \mathbb{R}^{2n} \to \mathbb{R}$ Hamiltonian,

the evolution eqns for $(x_1(t), \ldots y_n(t))$ are

$$\begin{cases} \dot{x}_j(t) = \partial H / \partial y_j \\ \dot{y}_j(t) = -\partial H / \partial x_j \end{cases}$$

Hamilton's eqns.

Globally on (M, ω), ω determines

$$TM \xrightarrow{\sim} T^*M$$
$$v \mapsto \omega(v, -)$$

so $H : M \to \mathbb{R} \rightsquigarrow dH \in \Omega^1(M)$

$\rightsquigarrow X_H \in \Gamma(TM)$ Hamiltonian v.f;

$(\iota_{X_H} \omega = dH)$.

Hamilton's eqns are $\dot{\underline{x}}(t) = X_H(\underline{x}(t))$,

so we are evolving along the flow $\varphi_H(t)$

of X_H.

<u>Key lemma</u> $\varphi_H(t)$ preserves ω.

<u>Proof</u> Infinitesimally,

$$\mathcal{L}_{X_H}\omega \underset{\nearrow}{=} d(\iota_{X_H}\omega) + \iota_{X_H}(\cancel{d\omega}) = ddH = 0.$$

Cartan's magic formula

$\square$

$\Rightarrow (M,\omega)$ has <u>a lot</u> of symmetries!

<u>Rmk</u> In general, a symmetry $\varphi : (M,\omega)^{\circlearrowleft}$ is called a <u>symplectomorphism</u>, these form the group $\mathrm{Symp}(M,\omega)$.

The flow of a vector field X preserves ω if $\iota_X\omega$ is closed; X is hamiltonian if $\iota_X\omega$ is exact $(= dH)$.

Can also do time dependent vector fields!

Let's discuss the geometric incarnation of Noether's principle (symmetry $\leftrightarrow$ conserved quantity). Consider an action of G Lie group on (M, ω), i.e. $\rho : G \to Symp(M, \omega)$ homomorphism. Let's assume:

(*) The differential at id_G has image in

$$\rho_* : \mathfrak{g} \to HamVect(M, \omega) \subseteq \Gamma(TM)$$

$\Rightarrow \forall \gamma \in \mathfrak{g}$, $\rho_*(\gamma) = X_{H_\gamma}$ for some $H_\gamma \in C^\infty(M)$, (well defined up to constant)

$\Rightarrow$ we obtain map $\mathfrak{g} \to C^\infty(M)$ linear;

dually, $\mu : M \to \mathfrak{g}^*$ s.t.

$$\langle \mu(x), \gamma \rangle = H_\gamma(x).$$

(**) $\mu : M \to \mathfrak{g}^*$ is G-equivariant.

<u>Def</u> If these hold, we say $G \curvearrowright (M,\omega)$ is <u>Hamiltonian action</u>, and $\mu : M \to \mathfrak{g}^*$ is <u>moment map</u>.

<u>Rule</u> There's general conditions to prove $G \curvearrowright (M,\omega)$ is Hamiltonian, we'll check it by hand in interesting cases.

<u>Ex</u> $M = T^*\mathbb{R}^3 = (\mathbb{R}^6, \omega_{std})$

If $G = \mathbb{R}^3 \curvearrowright M$ action induced by translations $\mathbb{R}^3 \curvearrowright \mathbb{R}^3$, then the moment map is $\mu(\underline{x},\underline{y}) = \underline{y} \in (\mathbb{R}^3)^* \cong \mathbb{R}^3$. This is the usual <u>momentum</u>!

If $G = SO(3) \curvearrowright M$ rotations on $\mathbb{R}^3$ ②

$\mu(\underline{x}, \underline{y}) = \underline{x} \times \underline{y} \in so(3)^* \equiv \mathbb{R}^3$ (angular momentum)

Here we use the identification of Lie algebras $(\mathbb{R}^3, \times) \xrightarrow{\sim} (so(3), [\cdot,\cdot])$

$$\underline{a} = (a_1, a_2, a_3) \longmapsto \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}.$$

plus $so(3) \cong so(3)^*$ ($SO(3)$-equivariantly)

Then $\rho : SO(3) \to Symp(T^*\mathbb{R}^3, \omega_{can})$ has

$$\underbrace{d\rho(\underline{a})}_{\text{vector field on } T^*\mathbb{R}^3} (\underline{x}, \underline{y}) = (\underline{a} \times \underline{x}, \underline{a} \times \underline{y}).$$

This is the Hamiltonian vector field of

$H_{\underline{a}} : T^*\mathbb{R}^3 \to \mathbb{R}$ $(\underline{x}, \underline{y}) \mapsto (\underline{x} \times \underline{y}) \cdot \underline{a} =$

$$= \langle \underbrace{\underline{x} \times \underline{y}}_{\mu}, \underline{a} \rangle$$

Ex 2 $U(n) \curvearrowright \mathbb{C}^n \equiv \mathbb{R}^{2n}$

(here $\omega_{std} \equiv \frac{i}{2} \sum_{j=1}^{n} dz_j \wedge d\bar{z}_j$)

use inner product $Tr(AB^*)$

Then $\mu : \mathbb{C}^n \to \mathfrak{u}(n)^* \cong \mathfrak{u}(n)$

$$\underline{z} \longmapsto -\frac{i}{2} \underline{z}\, \underline{z}^*$$

Ex 3 For $U(1) \curvearrowright \mathbb{C}^n$ the moment map is

$$\boxed{\mu(\underline{z}) = -\frac{i}{2} |\underline{z}|^2} \in i\mathbb{R} = \mathfrak{u}(1)^* .$$

will be important for us!

Rmk $\mu + c$ is also moment map $c \in i\mathbb{R}$!

Marsden-Weinstein (1974) quotients of symplectic mfld by $G \curvearrowright M$ Hamiltonian. Assume

cpt

•) $c \in \mathfrak{g}^*$ is G-invariant & regular value of μ

$\Rightarrow \mu^{-1}(c) \subseteq M$ smooth mfld with $G \curvearrowright \mu^{-1}(c)$.

•) $G \curvearrowright \mu^{-1}(c)$ freely, so $\mu^{-1}(c)/G$
is smooth mfld (of dim $M - 2\dim G$).

Then $\mu^{-1}(c)/G$ $(=: X/\!\!/G)$ admits
a natural symplectic form induced by ω
(it's called symplectic quotient)

Ex $U(1) \curvearrowright \mathbb{C}^n$, pick $c \neq 0$ in $i\mathbb{R}$.
$\Rightarrow \mu^{-1}(c)/U(1)$ is either ϕ, or $\mathbb{C}P^{n-1}$
(with a scaled version of ω_{FS}!!).

Key symplectic linear algebra observation:

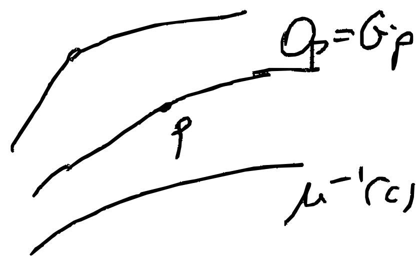
(V, ω) symplectic v.s. $(\cong (\mathbb{R}^{2n}, \omega_{std}))$

$I \subseteq V$ isotropic i.e $\omega|_I \equiv 0$
$(\Rightarrow I^\perp \supset I)$

Then ω induces natural symplectic form Ω on $I^\perp / I$ by $\Omega([v],[w]) = \omega(v,w)$.

The theorem follows by checking that $T_p\mathcal{O}_p$ is isotropic, and

$T_p\mu^{-1}(c)\ (= \ker d\mu_p) = T_p\mathcal{O}_p^\perp$



<u>Moment maps & gauge theory</u> key observations of Atiyah-Bott (1983) about surfaces.

$E \to \Sigma$ $\mathrm{rk}_{\mathbb{C}} = n$ hermitian bundle. U(n)-bundle.

$\mathcal{A} = \{$hermitian connections$\} = A_0 + \Omega^1(\mathfrak{g}_E)$

is naturally an ∞-dimensional symplectic manifold.

$T_A \mathcal{A} = \Omega^1(\mathfrak{g}_E)$, and

$\omega(a,b) := \int_{\Sigma^4} \mathrm{Tr}(a \wedge b)$ (constant $\Rightarrow$ closed)
$\Omega^2(M)$.

Now $\mathcal{G}$ = gauge group $\curvearrowright \mathcal{A}$ via

$$u \cdot A = A - (d_A u) u^{-1}.$$

$\mathrm{Lie}(\mathcal{G}) = \Omega^0(\mathfrak{g}_E)$, and the differential

is the map $\mathrm{Lie}(\mathcal{G}) \to \Gamma(T\mathcal{A})$

$$\phi \mapsto (A \mapsto -d_A\phi \in T_A\mathcal{A} = \Omega^1(\mathfrak{g}_E))$$

<u>Lemma</u> the vector field X sending $A \mapsto -d_A\phi$

is Hamiltonian for $H: \mathcal{A} \to \mathbb{R}$

$$A \mapsto \int \mathrm{Tr}(F_A \wedge \phi)$$

<u>Proof</u> $F_{A+ta} = F_A + t\, d_A a + O(t^2)$

$\Rightarrow dH_A(a) = \int \mathrm{Tr}(d_A a \wedge \phi)$

Now $\omega_A(X, a) = \int \mathrm{Tr}(-d_A\phi \wedge a)$ (by def)

$= \int \mathrm{Tr}(\phi \wedge d_A a)$ (by Stokes) $\square$

Using identification $\Omega^0(\mathfrak{g}_E)^* \equiv \Omega^2(\mathfrak{g}_E)$,

and gauge invariance properties of F_A, we have

<u>Thm</u> The action $\mathcal{G} \curvearrowright \mathcal{A}$ is Hamiltonian

with moment map

$$\mathcal{A} \to \Omega^2(\mathfrak{g}_E) \quad A \mapsto F_A$$

In particular, one expects spaces of

solutions <u>mod gauge</u> to be symplectic

manifolds in good circumstances!

<u>Rmk</u> this is all <u>formal</u>!

$\boxed{\text{Lecture 10}}$ Moduli spaces in gauge theory.

General principle sets of gauge theoretic objects (configurations or solutions) up to symmetry have natural structures.

Recap: $L \to (\Sigma, g)$ hermitian line bundle.

$\mathcal{A} = \{ A \text{ compatible connection} \}$

$\mathcal{C} = \{ (A, \Phi) \mid A \text{ compatible connection}, \Phi \in \Gamma(L) \}$

$\cup$

$\mathcal{C}^* = \{ (A, \Phi) \mid \Phi \not\equiv 0 \}$

$\cup$

$\mathcal{N} := \{ (A, \Phi) \mid \Phi \not\equiv 0, \bar\partial_A \Phi = 0 \}.$

$\cup$

$\mathcal{V} := \left\{ (A, \Phi) \mid \Phi \not\equiv 0 \; \begin{cases} \bar\partial_A \Phi = 0 \\ *F_A = \frac{i}{2}(|\Phi|^2 - 1) \end{cases} \right\}$

These are all acted on by

$$\mathcal{G} = \{u : \Sigma \to U(1)\}, \quad u\cdot(A,\Phi) = (A - u^{-1}du,\ u\cdot\Phi)$$

$$\Rightarrow \mathcal{A}/\mathcal{G},\ \mathcal{C}/\mathcal{G} \supseteq \mathcal{C}^*/\mathcal{G} \supseteq \mathcal{N}/\mathcal{G} \supseteq \underbrace{\mathcal{V}/\mathcal{G}}_{} = \mathcal{M}$$ want to understand this!

<u>Notice</u>: the action of $\mathcal{G} \curvearrowright \mathcal{C}^*$ is <u>free</u>

($u^{-1}du \equiv 0 \Rightarrow u$ constant), so one hopes that

$\mathcal{C}^*/\mathcal{G}$ is an infinite dimensional smooth mfld.

To make this precise, we work with

<u>Sobolev completions</u>. Fix $\bar{A} \in \mathcal{A}$,

$$\mathcal{A}_k := \bar{A} + L^2_k(i\Omega^1) \leftarrow L^2_k \text{ connections}.$$

$$\mathcal{C}_k = \{(A,\Phi) \mid A \in \mathcal{A}_k,\ \Phi \in L^2_k(\Gamma(L))\}$$

L^2_k configurations, Hilbert affine space.

For $k \geq 1$, consider

$$\mathcal{G}_{k+1} = \{ u : \Sigma \to U(1) \} \subseteq L^2_{k+1}(\Sigma, \mathbb{C})$$

(importantly, $\subseteq C^0(\Sigma, \mathbb{C})$, so u has well-defined values and preserves bundle topology!).

Then we have $\mathcal{G}_{k+1} \curvearrowright \mathcal{C}_k$ continuous

$$\Big(A - \underset{L^2_{k+1}}{\underset{\Downarrow}{u}}{}^{-1} \underbrace{du}_{\in L^2_k} \;,\; \underset{L^2_{k+1}}{\underset{\Downarrow}{u}} \cdot \Phi_{\in L^2_k} \Big),$$ in fact smooth.

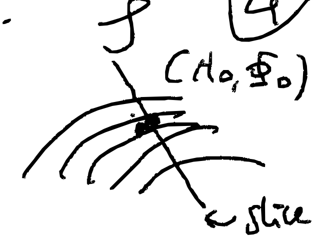
<u>Thm!</u> $\mathcal{C}^*_k / \mathcal{G}_{k+1}$ is a Hilbert manifold. $(k \geq 1)$.

The proof is analogous to: G cpt Lie group, $G \curvearrowright M$ freely $\Rightarrow M/G$ smooth mfld. Two steps

a) M/G is Hausdorff

b) build local charts for M/G

We'll work out a) in detail.

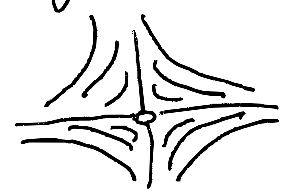


Rmk b) follows by considering the analogue of Coulomb gauge fixing:

$S := L^2$ orthogonal to $T_{(A_0,\Phi_0)}(\mathcal{G}\cdot(A_0,\Phi_0))$.

a) already fails in finite dim if G not cpt! e.g. $\mathbb{R}^{>0} \curvearrowright \mathbb{R}^2 \setminus (0,0)$

$$t\cdot(x,y) = (tx, t^{-1}y)$$



In our setup, we want to show that

if $(A_n, \Phi_n) \to (A_\infty, \Phi_\infty)$, and for $u_n \in \mathcal{G}_{U(1)}$

$(A_n', \Phi_n') = u_n(A_n, \Phi_n) \to (A_\infty', \Phi_\infty')$

$\Rightarrow \exists\, u_\infty$ s.t. $u_\infty(A_\infty, \Phi_\infty) = (A_\infty', \Phi_\infty')$.

First, the class $[u_n] \in \pi_0 \mathcal{G}_{U(1)}$ is eventually constant, this is represented by as

$$\left[\frac{1}{2\pi i} u_n^{-1} du_n\right] \in H^1(\Sigma;\mathbb{Z}) \quad (= \pi_0(\mathcal{G}_{U(1)}))$$

Fix 1-forms $\{\beta_i\}$, $i=1,\dots 2g$ with

of $H^1(\Sigma;\mathbb{Z})$. Then

$$\int_\Sigma \beta_i \wedge u_n^{-1} du_n = \int \beta_i \wedge (A_n - A_n') \to \int \beta_i \wedge (A_\infty - A_\infty')$$

$\Rightarrow$ can assume $u_n \in$ identity component, $u_n = e^{\xi_n}$.

Write $\xi_n = \underset{\text{constant}}{\xi_n^0} + \xi_n^\perp \in \langle\text{constants}\rangle^\perp$.

$\Rightarrow d(\xi_n^\perp) = A_n - A_n'$ is Cauchy in L^2_k

$\Rightarrow \xi_n^\perp$ is Cauchy in L^2_{k+1} (Gårding)

$\Rightarrow$ converges to $\xi_\infty^\perp$. Choose subsequence s.t. $e^{\xi_n^0} \in U(1)$ converges, then

$u_n = e^{\xi_n^0} \cdot e^{\xi_n^\perp} \to u_\infty$, gauge transformation

we were looking for. $\square$

$\mathcal{C}_k$ is a good space to study our PDEs! For example we have continuous (indeed smooth) map

$$\mathcal{F} : \mathcal{C}_k \longrightarrow L^2_{k-1}(\Omega^{0,1}(L))$$
$$(A, \Phi) \longmapsto \bar{\partial}_A \Phi .$$

Indeed, writing $A = A_0 + a$, $a \in L^2_k(i\Omega)$, it's

$$\bar{\partial}_{A_0} \Phi + a^{0,1} \cdot \Phi . \quad (\bar{\partial}_{A_0} \text{ has smooth coeff!})$$

If $k \geq 2$, this is in L^2_{k-1} by Sobolev multiplication. If $k = 1$, we use $\forall p < \infty$ $L^2_1 \hookrightarrow L^p$ (in fact, it's cpt!) ($L^2_1 \not\hookrightarrow C^0$!)

so $a^{0,1} \cdot \Phi \in L^4 \cdot L^4 \subseteq L^2$ by Cauchy-Schwarz.

Notice $\mathcal{N}_k = \{(A, \Phi) \mid \Phi \neq 0, \bar{\partial}_A \Phi = 0\}$

$$= \mathcal{F}^{-1}(0) \cap \mathcal{C}_k^*$$

$\hookrightarrow$ no- action.

Let's compare with the holomorphic picture.

Recall correspondence for L:

$$\{\text{hol structures } \mathcal{L}\} \xleftrightarrow{1:1} \{\text{hermitian connections } A\}$$

$$\mathcal{A}^{0,1} = \{ d^{(0,1)} : \Omega^0(L) \to \Omega^{0,1}(L) \text{ satisfying } (0,1)\text{-Leibniz}\}.$$

$\mathcal{L} \mapsto$ unique A s.t. $\bar\partial_A = \bar\partial_{\mathcal{L}}$.

This is an affine isomorphism via the map

$$L^2_k(\Omega^{0,1}) \xleftrightarrow{\sim} L^2_k(i\Omega^1)$$

$$a^{0,1} \longleftarrow\!\shortmid\; a$$

($\mathbb{C}$-linear for action of $*$ on Ω^1).

The natural symmetries of $\mathcal{A}^{0,1}$ are

$$\mathcal{G}^{\mathbb{C}}_{k+1} = \{ g : \Sigma \to \mathbb{C}^* \}.$$

$g \cdot d^{(0,1)} = d^{(0,1)} - g^{-1}\bar\partial g$; at the level of A it's

$$A \mapsto A - (g^{-1}\bar\partial g) + \overline{(g^{-1}\bar\partial g)}$$

<u>Thm 2</u> $\mathcal{A}^{0,1}_k / \mathcal{G}^{\mathbb{C}}_{k+1} \cong H^1(\Sigma;\mathbb{R}) / H^1(\Sigma;\mathbb{Z})$

$\shortparallel$

{hol structures $\mathcal{L}\}$ / iso

$\uparrow$ $2g(\Sigma)$-dim torus.

(<u>Picard torus.</u>)

<u>Proof</u> $\pi_0(\mathcal{G}^{\mathbb{C}}_{k+1}) = H^1(\Sigma;\mathbb{Z})$, so

$$\mathcal{A}^{0,1}/\mathcal{G}^{\mathbb{C}} = \left(\mathcal{A}^{0,1}/(\mathcal{G}^{\mathbb{C}})^{id}\right) / H^1(\Sigma;\mathbb{Z}).$$

g in identity component is $g = e^f$ for $f : \Sigma \to \mathbb{C}$,

acting via $e^f \cdot A = A - \bar{\partial} f + \partial \bar{f}$.

<u>Computation</u>: if $f = u + iv$, this is

$$A - i(dv - * du)$$

$\Rightarrow$ can add any exact / coexact form

$\Rightarrow \mathcal{A}^{0,1}/(\mathcal{G}^{\mathbb{C}})^{id} \cong \mathcal{H}^1$ harmonic 1-forms

$\square$.

<u>Cor of proof</u> every $A \in \mathcal{A}_k$ is $\mathcal{G}^{\mathbb{C}}_{k+1}$ equivalent to a smooth connection.

<u>Rmk</u> not true for $\mathcal{G}_{k+1}$! Curvature is $\mathcal{G}_{k+1}$ invariant.

Now, we also have $\mathcal{G}^{\mathbb{C}}_{k+1} \curvearrowright \mathcal{N}_k$, and every (A, Φ) is equivalent to a smooth one: A is equivalent to A^{smooth}, and $\ker(\bar\partial_{A^{\text{smooth}}}) \subseteq C^\infty$ by elliptic reg.

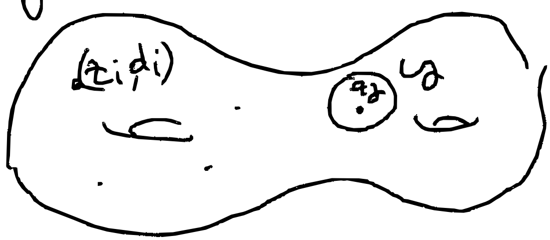
<u>Thm 3</u> $\mathcal{N}_k / \mathcal{G}^{\mathbb{C}}_{k+1} \overset{1:1}{\longleftrightarrow} \mathrm{Sym}^d(\Sigma)$. ($d$ ~ degree of L)

$(A, \Phi) \longmapsto$ zeroes of Φ (with multiplicity).

<u>Rmk</u> from this, to identify vortex moduli

space we need to show that each orbit of $\mathcal{G}^a \curvearrowright \mathcal{N}$ contains a solution of the 2nd vortex eqn, unique up to $\mathcal{G}$

Next time! ↳ ⚠ not $\mathcal{G}^a$ invariant!



Proof Surjectivity:

Choose trivial bundles on $S \setminus \{z_1, \ldots\}$, $U_j \cong D^2$, and use transition functions

$$D^2 \setminus \{0\} \to \mathbb{C}^* \quad z \mapsto z^{d_j},$$

the constant section 1 on $S \setminus \{z_1, \ldots\}$ extends with zero of order d_j.

Injectivity If (A, Φ), $(A', \Phi') \in \mathcal{N}_k$ are such that Φ, Φ' have the same zeroes with multiplicities, then $\exists!$

$g \in \mathcal{G}^{\mathbb{C}}_{k+1}$ s.t $g \cdot \Phi = \Phi'$.

Then $g \cdot A = A'$ (check!) $\square$.

Rmk $\mathcal{N}/\mathcal{G}^{\mathbb{C}} \xrightarrow{\text{forget } \Phi} \mathcal{A}/\mathcal{G}^{\mathbb{C}}$ is the Abel-Jacobi map.

Going back to $\mathcal{N}_k$, we have

<u>Thm</u> $\mathcal{N}_k$ is a Hilbert mfld. $(k \geq 1)$

<u>Rmk</u> useful because we'll do symplectic quotient!

<u>Proof</u> We'll apply implicit function thm at

(A, Φ) with $\bar{\partial}_A \Phi = 0$, $\Phi \not\equiv 0$.

$$D\mathcal{F}_{(A,\Phi)} : L^2_k(i\Omega^2) \oplus L^2_k(\Gamma(L)) \to L^2_{k-1}(\Omega^{0,1}(L))$$

$$(a, \phi) \mapsto \bar{\partial}_A \phi + a^{0,1} \Phi.$$

We need to prove <u>surjectivity</u>.

By $\mathcal{G}^{\mathbb{C}}$ invariance ($\bar\partial_{g\cdot A}(g\Phi) = g\bar\partial_A\Phi$), we can assume (A,Φ) smooth.

Note $\operatorname{Im} D\mathcal{F}_{(A,\Phi)} \supseteq \bar\partial_A\phi \Rightarrow$ finite codim $\Rightarrow$ closed; if it's not everything, $\exists\ b^{0,1} \in L^2_{k-1}(\Omega^{0,1}(L))$ L^2-orthogonal to image, i.e

$$\langle \bar\partial_A\phi + a^{0,1}\Phi, b^{0,1}\rangle_{L^2} = 0 \quad \forall a, \phi.$$

Using $(0,\phi)$, we see $\bar\partial_A^* b^{0,1} = 0$

weakly. $\Rightarrow b^{0,1}$ smooth. Now

$\bar\partial_A(\bar * b^{0,1}) = 0$ for $\bar\partial_A : \Omega^{1,0}(L) \to \Omega^{1,1}(L)$

$\Rightarrow b^{0,1}$ is not identically vanishing on dense

open set $\Rightarrow \exists\ a^{0,1}$ s.t

$\langle a^{0,1}\Phi, b^{0,1}\rangle_{L^2} \neq 0$ $\square$.

Lecture 11 Vortices & symmetric products

(Σ, g) cpt Riemannian surface, $L \to \Sigma$ hermitian line bundle of degree $d = d(L) \geq 0$.

$$\mathcal{M} = \left\{ (A, \Phi) \mid \Phi \not\equiv 0, \begin{cases} \bar\partial_A \Phi = 0 \\ * F_A = \frac{i}{2}(|\Phi|^2 - 1) \end{cases} \right\} \Big/ \mathcal{G}$$

<u>vortex moduli space.</u>

<u>Recall</u> $\deg(L) = \int_\Sigma \frac{i}{2\pi} F_A = \frac{1}{4\pi} \int_\Sigma (1 - |\Phi|^2)\, d\mathrm{vol} < \frac{\mathrm{Area}(\Sigma)}{4\pi}$.

<u>Thm</u> (Taubes, Bradlow, García-Prada)

Assume $d < \mathrm{Area}(\Sigma)/4\pi$. Then

$\mathcal{M}$ admits a natural structure of smooth symplectic mfld of $\dim = 2d$

$$\mathcal{M} \xrightarrow{1:1} \mathrm{Sym}^d(\Sigma) \quad [(A, \Phi)] \to \{\text{zeroes of } \Phi\}$$

is a homeomorphism.

This will follow by comparing two geometric views: 2

<u>Symplectic</u>: $\mathcal{N} = \{ \Phi \neq 0, \bar{\partial}_A \Phi = 0 \} \subseteq \mathcal{C}^*$

is a <u>symplectic</u> submanifold (∞-dim)

(will elaborate more later!)

$\mathcal{G} \curvearrowright \mathcal{N}$ is hamiltonian action with moment map

$$\mu : \mathcal{N} \to i\Omega^0 \quad (A, \Phi) \mapsto * F_A - \frac{i}{2} |\Phi|^2 + \frac{i}{2} .$$

$\Rightarrow M = \mu^{-1}(0) / \mathcal{G} = \mathcal{N} /\!\!/ \mathcal{G}$ symplectic quotient!

<u>Complex</u>: $\mathcal{N} / \mathcal{G}^{\mathbb{C}} \overset{\text{last time}}{\overset{?}{=}} \mathrm{Sym}^d(\Sigma)$.

<u>Upshot</u> $\forall (A, \Phi) \in \mathcal{N}$, need to show

$\exists\, g \in \mathcal{G}^{\mathbb{C}}$ s.t $\mu(g \cdot (A, \Phi)) = 0$

(unique up to $u \in \mathcal{G}$), i.e

the value $\mu = 0$ is attained in each (3)

orbit. Let's focus on existence first (harder!)

This will be based on special properties

in the case of X Kähler, i.e X cplx

mfld (with induced $J : TX^{\circlearrowleft}$ a.c.s)

equipped with ω symplectic compatible:

$g(v,w) := \omega(v, Jv)$ is Riemannian metric.

Ex.) $(\mathbb{C}^n, \omega_{std})$, $(\mathbb{C}P^n, \omega_{FS})$

•) (X,ω) Kähler $V \subseteq X$ cplx submanifold

$\Rightarrow (V, \omega|_V)$ is Kähler.

•) $\mathcal{N} \subset \mathcal{C}^*$ is Kähler submanifold

(in suitable Sobolev completions).

The key computation (in finite dim)

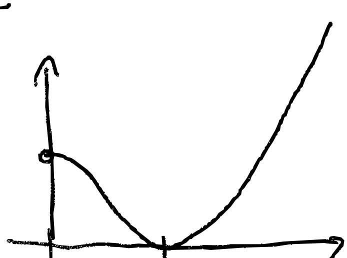
<u>Thm</u> (Fraenkel '59, simplest case $G = \mathbb{C}^*$)

(X, ω) Kähler, $\mathbb{C}^* \curvearrowright X$ freely by biholomorphisms s.t. the restriction $U(1) \curvearrowright X$ is Hamiltonian action (with moment map $\mu : X \to i\mathbb{R} = \mathrm{Lie}(U(1))$).

Consider $F : X \to \mathbb{R}^{\geq 0}$ $x \mapsto |\mu(x)|^2$.

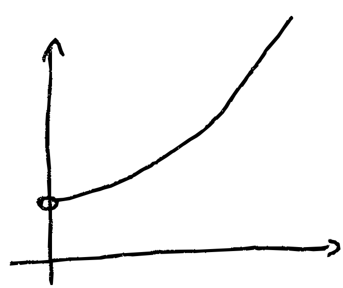
Then the critical points of F are critical points of $F|_{\mathbb{C}^*\text{-orbits}}$, and $\mu = 0$ at them.

<u>Ex</u> $\mathbb{C}^* \curvearrowright \mathbb{C}^n \setminus \{0\}$, $\mu = -\frac{i}{2}|\underline{z}|^2 + ic$ $c \in \mathbb{R}$.



$|\underline{z}|$

$c > 0$



$|\underline{z}|$

$c < 0$

<u>Rmk</u> setup $\Rightarrow$ $U(1)$ acts by isometries

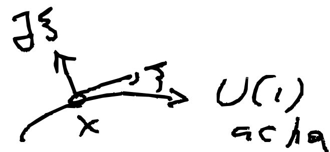
<u>Proof</u> Let's identify $i\mathbb{R} \cong \mathbb{R}$, so $\mu : X \to \mathbb{R}$.

$$\langle \operatorname{grad} F, w \rangle_x = dF_x(w) = 2\,\mu_x\, d\mu_x(w).$$

By def, $\mu(x) = \langle \mu(x), 1 \rangle$ is the Hamiltonian function for the vector field ξ generating $U(1) \curvearrowright X$, i.e.

$$d\mu = \omega(\xi, -)$$

$$\Rightarrow d\mu_x(w) = \omega(\xi, w) = \omega(J\xi, Jw) = \langle J\xi, w \rangle$$



$$\Rightarrow \operatorname{grad} F_x = 2\mu_x \cdot J\xi \in T_xX$$

$\Rightarrow$ grad F is tangent to $\mathbb{C}^*$ orbits. Action free $\Rightarrow \xi \neq 0$ everywhere $\Rightarrow$ at critical points $\mu = 0$. $\square$

To apply in our setup, notice that

if $\mu(A,\Phi)=0 \Rightarrow \|\Phi\|^2_{L^2} = \text{Area}(\Sigma) - 4\pi d$

is determined. So we can consider

$$\overline{\mathcal{N}} := \{\bar\partial_A \Phi = 0,\ \|\Phi\|^2_{L^2} = \text{Area}(\Sigma) - 4\pi d\} / \underset{\text{constants}}{U(1)}$$

(non empty by assumption!)

$\overline{\mathcal{G}}^{\mathbb{C}} = \mathcal{G}^{\mathbb{C}} / \mathbb{C}^* \leftarrow$ constants $\curvearrowright \overline{\mathcal{N}}$ where

$\bar g \cdot (A,\Phi) := g\cdot(A,\Phi)$ where $g \in \mathcal{G}^{\mathbb{C}}$ is s.t

$\|g\Phi\|_{L^2} = \|\Phi\|_{L^2}$ (unique, up to $U(1)$)

<u>Rmk</u> from a <u>symplectic</u> viewpoint, we have

$U(1) \subset \mathcal{G}$ and this is equivalent to:

$$\mathcal{M} = \mathcal{N}/\!/\mathcal{G} = \underset{\overline{\mathcal{N}}}{\underset{\shortparallel}{\left(\mathcal{N}/\!/U(1)\right)}} /\!/ \left(\mathcal{G}/U(1)\right)$$

<u>Goal</u> Fix $(A_0, \Phi_0) \in \mathcal{N}$, show that

$\|\mu\|^2_{L^2}$ attains a minimum at (A_{min}, Φ_{min})

$\in \mathcal{O} := \overline{\mathcal{G}^{\mathbb{C}}} \cdot (A_0, \Phi_0)$ $(\Rightarrow \mu(A_{min}, \Phi_{min}) = 0)$

This we can do with a <u>direct</u> variational approach: Let

$$C = \inf_{\mathcal{O}} \|\mu(A, \Phi)\|^2_{L^2}, \text{ and consider}$$

$(A_n, \Phi_n) (= \bar{g}_n (A_0, \Phi_0), \bar{g}_n \in \mathcal{G}^{\mathbb{C}})$ s.t

$$\|\mu(A_n, \Phi_n)\|^2_{L^2} \searrow C.$$

<u>Want</u> up to subsequence & $\mathcal{G}$

$(A_n, \Phi_n) \to (A_{min}, \Phi_{min}) \in \mathcal{O}$ realizing inf.

<u>Notice</u> $\|\mu(A_n, \Phi_n)\|^2_{L^2} \leq C. \ \forall n.$

$\boxed{\text{Step 1}}$ $\|F_{A_n}\|^2_{L^2}$ is bounded.

Proof: This follows from key identity

$$\frac{1}{2}\int |F_A|^2 + \frac{1}{2}\int |\nabla_A \Phi|^2 + \frac{1}{8}\int (1-|\Phi|^2)^2 =$$

$$= \cancel{\|\bar\partial_A \Phi\|^2_{L^2}} + \frac{1}{2}\|\mu(A,\Phi)\|^2_{L^2} + \pi d.$$

in our setting.

$\boxed{\text{Step 2}}$ up to $\mathcal{G}$, $\{A_n\}$ is bounded in L^2_1.

(ie $A_n = A_0 + a_n$, $\|a_n\|_{L^2_1} < C$ $\forall n$).

Proof: we can make transformations

$A_n \to u_n \cdot A_n$, $u_n \in \mathcal{G}$ so that

•) A_n is in Coulomb gauge w.r.t to A_0

$d^* a_n = 0$ (just need $\mathcal{G}_{id}$)

∘) $a_n^{harm} \in \mathcal{H}^1$ is in fixed fundamental domain

for $H^1(\Sigma;\mathbb{R})/H^1(\Sigma;\mathbb{Z})$. $\boxed{///}$

Then F_{A_n} bounded $\Rightarrow$ da_n bounded

$\Rightarrow$ L^2_1-bounds by Gårding ineq.

$\boxed{\text{Step 3}}$ $\{\Phi_n\}$ is bounded in L^2_1

<u>Proof</u> By key identity, $\{\Phi_n\}$ is bounded in L^4.

Now $\bar\partial_{A_n} = \bar\partial_{A_0} + a_n^{0,1}$, so

$$\|\Phi_n\|_{L^2_1} \leq C(\|\bar\partial_{A_0}\Phi_n\|_{L^2} + \|\Phi_n\|^2_{L^2}) \text{ (Gårding)}$$

$$= C(\|a_n^{0,1}\cdot\Phi_n\|_{L^2} + \|\Phi_n\|^2_{L^2})$$

$\hookrightarrow$ fixed!

this is bounded because $L^2_1 \hookrightarrow L^4$

by Sobolev embedding in dim 2.

$\boxed{\text{Step 4}}$ Abstract non sense moment:

<u>Prop</u> H cplx separable Hilbert space, $\{x_i\} \subseteq H$ bounded sequence.

$\Rightarrow$ after taking subsequence,

$x_i \rightharpoonup x \in H$ <u>weak</u> convergence, i.e

$\langle x_i, v \rangle_H \to \langle x, v \rangle_H \quad \forall\, v \in H$.

<u>Ex</u> In $\ell^2(\mathbb{N})$, $e_n = (0, \dots 0, 1, 0, \dots)$
nth place

$e_n \rightharpoonup 0$.

<u>Consequence</u> $\exists\, (A, \Phi) \in \mathcal{C}_1$ ← L^2_1-configurations

s.t $(A_n, \Phi_n) \rightharpoonup (A, \Phi)$ (weak L^2_1)

This is our candidate minimum in $\mathcal{B}$!!

$\boxed{\text{Step 5}}$ $\|\mu(A, \Phi)\|_{L^2}$ is $c = \inf$

This is because the embedding $L^2_1 \hookrightarrow L^4$ (11)

is actually cpt (cf. Rellich), and

cpt ops send weakly convergent sequences

to convergent ones.

To conclude <u>existence</u> proof.

$\boxed{\text{Step 6}}$ (A, Φ) is in the orbit

$\overline{\mathcal{G}^{\mathbb{C}}_2} \cdot (A_0, \Phi_0)$.

<u>Idea</u>: write $(A_n, \Phi_n) = \bar{g}_n \cdot (A_0, \Phi_0)$, show

$\bar{g}_n$ converges in L^2_2 as in proof of Hausdorff

$\boxed{\text{Step 7}}$ (A, Φ) is smooth $\in \overline{\mathcal{G}^{\mathbb{C}}} \cdot (A_0, \Phi_0)$.

<u>Idea</u> elliptic bootstrap in Coulomb gauge

(need <u>serious</u> L^p theory for this step!!).

To prove <u>uniqueness</u> up to action of $\mathcal{G}$

notice $\mathcal{G}^{\mathbb{C}}/\mathcal{G} \cong \{e^h \mid h:\Sigma \to \mathbb{R}\}$,

and we computed last time

$e^h\cdot(A,\Phi) = (A + i * dh,\ e^h\Phi)$. If

$$\mu(e^h(A,\Phi)) = *F_A + *\underbrace{i d*dh}_{-\Delta h} - \frac{i}{2}e^{2h}|\Phi|^2 + \frac{i}{2} = 0$$

and $\mu(A,\Phi)=0$, subtracting we have

$$\Delta h = (1-e^{2h})\underbrace{|\Phi|^2}, \text{ has isolated zeroes}$$

Now, $\|e^h\cdot\Phi\|_{L^2} = \|\Phi\|_{L^2} \Rightarrow$ if $h \not\equiv 0$,

$h(\bar{x})>0$ at maximum $\bar{x}$ of h. But

$\Delta h(\bar{x}) \geq 0$ (in $\mathbb{R}^2$, $\Delta = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$ 

)

$\Rightarrow$ contradiction if $\Phi(\bar{x}) \neq 0$. If $\Phi(\bar{x})=0$,

use $\Delta h \leq 0$ near $\bar{x}$ + strong maximum principle $\square$.

$\boxed{\text{Lecture 12}}$ Seiberg-Witten theory

We have seen throughout the class the importance of first order elliptic PDEs ($d+d^*$, d^++d^* in 4D, $\bar\partial$ in 2D).

Dirac (1928): need first order operator D on $\mathbb{R}^n$ s.t $D^2=\Delta$ $\left(=-\frac{\partial^2}{\partial x_1^2}-\dots\frac{\partial^2}{\partial x_n^2}\right)$.

Assume $D=a_1\frac{\partial}{\partial x_1}+\dots+a_n\frac{\partial}{\partial x_n}$, constant coeff in some alg. structure (to be determined).

$$D^2=\Delta \iff \begin{cases} a_i^2=-1 \\ a_ia_j=-a_ja_i \text{ if } i\neq j. \end{cases}$$

$\boxed{n=1}$ Can take $a_1=i\in\mathbb{C}$,

$D=i\frac{\partial}{\partial x_1}: C^\infty(\mathbb{R};\mathbb{C})$ ↺

$\boxed{n=2}$ $a_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

$$D = a_1 \frac{\partial}{\partial x_1} + a_2 \frac{\partial}{\partial x_2} = \begin{bmatrix} 0 & -\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} & 0 \end{bmatrix}$$

acting on $C^\infty(\mathbb{R}^2; \mathbb{C}^2)$.

Rmk holomorphic/antiholomorphic derivatives!

$\boxed{n=3}$ $a_i = \sigma_i$ Pauli matrices (basis of $su(2)$)

$$\sigma_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$D \curvearrowright C^\infty(\mathbb{R}^3, \mathbb{C}^2)$.

$\boxed{n=4}$ We can consider the 4×4 block matrices

$$\begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & -\sigma_i^* \\ \sigma_i & 0 \end{bmatrix} \quad i = 1,2,3. \quad (*)$$

$D \curvearrowright C^\infty(\mathbb{R}^4, \mathbb{C}^4)$.

Continue inductively! We obtain

<u>Dirac operator</u> $D : C^\infty(\mathbb{R}^n; \mathbb{C}^{2^{\lfloor \frac{n}{2} \rfloor}})$

<u>Rmk</u> in general, the "algebraic structure" the a_i live in is the <u>Clifford algebra</u>, and we look at functions with values in a <u>Clifford module</u>; we obtained the "minimal" one, but for example one can also obtain $D = d + d^* : C^\infty(\mathbb{R}^n; \Lambda^* \mathbb{R}^n)$.

We'll discuss how to use this to define the <u>Seiberg-Witten invts</u> ('94) of X cpt oriented

<u>Sample application</u>: $\exists \{X_i\}_{i \in \mathbb{N}}$ collection all homeomorphic to each other, but not pairwise diffeomorphic.

<u>Def</u> (X^4, g) Riemannian, a <u>spin^c structure</u> is $\mathfrak{s} = (S, \rho)$ where

•) $S \to X$ is $rk_{\mathbb{C}} = 4$ hermitian, <u>Spinor bundle</u>

•) $\rho : TX \to su(S)$ <u>Clifford multiplication</u>

looks pointwise like $(*)$

<u>Rmk</u> not trivial <u>at all</u> that $\mathfrak{s}$ exists!!

$\{\text{spin}^c \text{ structures}\}/_{\text{iso}}$ is affine $/ H^2(X; \mathbb{Z})$

Define on T^*X by duality, extend via

$$\rho(\alpha \wedge \beta) = \frac{1}{2}\left(\rho(\alpha)\rho(\beta) + (-1)^{|\alpha||\beta|}\rho(\beta)\rho(\alpha)\right).$$

to $\rho : \Omega^*(X) \otimes \mathbb{C} \to \mathrm{End}(S)$

<u>Then</u> $\rho(\mathrm{dvol}_g) = \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix} \Rightarrow S = \underbrace{S^+ \oplus S^-}_{rk_{\mathbb{C}}=2}$

($\rho(v)$ for $v \in TX$ exchanges them).

We will consider pairs (A, Φ) where

- $\Phi \in \Gamma(S^+)$ is a (positive) spinor
- ∇_A is a <u>spin^c connection</u>, i.e. hermitian connection on S compatible with ρ & Levi-Civita

$$\nabla_A(\rho(X)\cdot\sigma) = \rho(\nabla X)\cdot\sigma + \rho(X)\cdot\nabla_A\sigma.$$

$$\forall X \in \Gamma(TX), \ \sigma \in \Gamma(S).$$

<u>Facts</u>
- ∇_A preserves splitting $S^+ \oplus S^-$
- $\nabla_A - \nabla_{A'} = a \otimes \mathbb{1}_S$ for $a \in i\Omega^1(X)$ (because ρ acts irreducibly).

$\Rightarrow \mathcal{C} = \mathcal{C}(K, S) = \{(A, \Phi)\}$ = affine$/ i\Omega^1 \oplus \Gamma(S^+)$.

configuration space. Gauge group = symmetries of (S, ρ) is

$\mathcal{G} = \{u : X \to U(1)\}$, via usual action

$u\cdot(A,\Phi) = (A - u^{-1}du,\ u\cdot\Phi)$: action

on $\mathcal{C}^* = \{(A,\Phi) \mid \Phi \neq 0\}$ is free.

↳ irreducible configurations.

$\boxed{\text{Eqn 1}}$ $\Gamma(S^+) \xrightarrow{\nabla_A} \Gamma(T^*X \otimes S^+) \xrightarrow{\rho} \Gamma(S^-)$

D_A^+ Dirac operator (1st order, elliptic)

(SW1) $D_A^+\Phi = 0 \quad (\in \Gamma(S^-))$.

$\boxed{\text{Eqn 2}}$ A connection on $S^+ \rightsquigarrow A^t$ connection on

$\det S^+ = \Lambda^2 S^+$ ← hermitian line bundle,

$F_{A^t} \in i\Omega^2(X)$.

Rmk If $\tilde{A} - A = a \otimes 1_S \Rightarrow \tilde{A}^t - A^t = 2a$.

traceless hermitian

Given $\Phi \in \Gamma(S^+)$, we have $(\Phi\Phi^*)_0 \in i\mathfrak{su}(S^+)$

(if $\Phi = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, $(\Phi\Phi^*)_0 = \begin{bmatrix} \frac{1}{2}(|\alpha|^2 - |\beta|^2) & \alpha\bar{\beta} \\ \bar{\alpha}\beta & \frac{1}{2}(|\beta|^2 - |\alpha|^2)\end{bmatrix}$)

Key point $\rho : \Lambda^+ \xrightarrow{\sim} su(S^+)$ (check)
↳ self-dual forms

Fix $\omega \in i\Omega^2$. The 2nd SW eqn (perturbed by ω) is

(SW2) $\frac{1}{2}\rho(F_{A^t}^+ - 4\omega^+) = (\Phi\Phi^*)_0 \in \Gamma(isu(S^+))$.

Def $\mathcal{M}_{(g,\omega)} = \{\text{solns to SW1 \& 2}\}/\mathcal{G}$. (depends on s!)

Rmk These are called Seiberg-Witten monopoles

because of the topological bound:

$$\int |F_{A^t}^+| + 4\int |D_A\Phi|^2 + \int\left(|\Phi|^2 + \frac{r}{2}\right)^2 \geq \int \frac{r^2}{4} - 4\pi^2 c_1^2(s)$$

where: •) r is the scalar curvature of g

•) $c_1(s) = c_1(S^+)$.

This follows from Chern-Weil + integrating by parts + Weitzenböck formula.

<u>Thm A</u> If $b_2^+ \geq 1$, for generic (g, a)

$\mathcal{M}_{(g,a)}$ is a smooth closed oriented mfld of dim

$$d(X,s) = \frac{c_1^2(s) - 2\chi(X) - 3\sigma(X)}{4} \in \mathbb{Z}.$$

This uses Atiyah-Singer index thm.

<u>Rmk</u> Key point: generically when $b_2^+ \geq 1$, all solutions are <u>irreducible</u>.

<u>Thm B</u> (simplest case) If $b_2^+ \geq 2$, $d(X,s) = 0$

$\Rightarrow \# \mathcal{M}_{(g,a)}$ independent of (g, a)

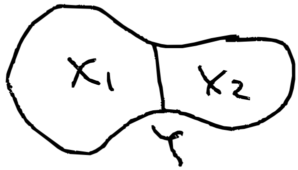
$\Rightarrow SW(X,s) \in \mathbb{Z}$. is diffeo invariant.

<u>Rmk</u> First such invts were defined by

Donaldson ('87) using the SU(2)

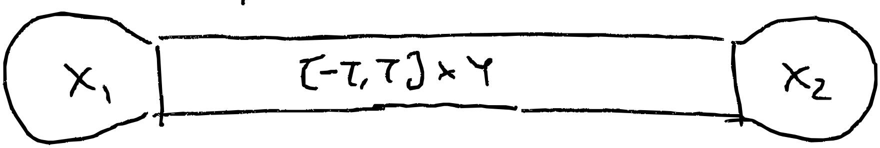
ASD eqns. $\{F_A^+ = 0\}/\mathcal{G}$ ← instanton moduli space, <u>non</u> cpt!

<u>Major issue</u> extremely hard to compute!

<u>Ex</u> $X =$  $\Rightarrow X' = X_1 \cup_\varphi X_2$ with $\varphi : Y \circlearrowleft$ diffeo

<u>Q</u> what's the effect on SW invts?

<u>Idea</u> exploit metric independence to "decouple" the two pieces via neck stretching



Now there's a 3D story of $spin^c$ structure $S_Y \to Y$ $rk_{\mathbb{C}} = 2$ hermitian; we have

$\mathcal{C}_Y = \{(B, \Psi)\}$. configuration space of Y.

On $I \times Y$, we have naturally

$$S^+ \underset{\substack{\nearrow \\ \text{via } \rho(\frac{\partial}{\partial t})}}{\cong} S^- \cong S_Y .$$

So a path $(B(t), \Psi(t))$ in $\mathcal{C}_Y$ gives

to $(A, \Phi) \in \mathcal{C}$ $\left[\nabla_A = \frac{d}{dt} + \nabla_{B(t)} \right]$.

$\boxed{\text{Key computation}}$ (A, Φ) solves SW eqns

$$\Updownarrow$$

$$\begin{cases} \frac{d}{dt} B = -\left(\frac{1}{2} * F_{B^t} + \rho^{-1}(\Psi\Psi^*)_0\right) \otimes 1_S \\ \frac{d}{dt}\Psi = -D_B \Psi \end{cases}$$

where $D_B : \Gamma(S)^{\circlearrowleft}$ 3D Dirac operator.

Further, these can be written as

$$\frac{d}{dt}(B, \Psi) = -\operatorname{grad} \mathcal{L}(B, \Psi)$$

where, fixing a base connection B_0, we have

$$\mathcal{L}(B, \Psi) = -\frac{1}{8} \int_Y (B^t - B^t_0) \wedge (F_{B^t} + F_{B^t_0}) + \frac{1}{2} \int_Y \langle D_B \Psi, \Psi \rangle \, dvol$$

is the <u>Chern-Simons-Dirac functional</u>.

Employing the ideas of Floer ('80s) for (formally) doing Morse theory in certain (very specific!) ∞ dim setups (and taking into account $U(1)$-action):

<u>Kronheimer–Mrowka</u> ('07) <u>monopole Floer homology groups</u> $HM(Y, \mathfrak{s}_Y)$;

(Roughly speaking, this is the homology of a chain complex generated by soln to 3D SW eqns $= \{\operatorname{grad}(B,\Psi)=0\}/\mathcal{G}_Y$)

These are where invariants of 4-mfld

with ∂Y take value (cf: Atiyah's TQFT framework).

These 3 mfld invts are still extremely difficult to understand / compute!

Going further down we look at $\Sigma \subset Y$.

$\boxed{\text{Key computation}}$ Translation invariant solns to $\mathrm{grad}(B,\Psi)=0$ on $I\times\Sigma$

$\Leftrightarrow$ solutions to (small variation) on vortex eqns on Σ.

<u>Heuristic</u> one can hope to define interesting invts of 3-mflds ($\cong HM(Y)$) by studying $\underbrace{\mathrm{Sym}^d(\Sigma)}$ $\to$ vortex moduli space, in particular its <u>symplectic geometry</u> (again using Floer's ideas)

<u>Ozsváth-Szabó</u> ('04) construction of Heegaard Floer homology.

$\hookrightarrow \cong HM$, but much easier to compute!